Physica B 457 (2015) 78–81

Contents lists available at ScienceDirect

Physica B

journal homepage: www.elsevier.com/locate/physb

Excitation of time-dependent quantum systems: An application of time–energy uncertainty relations

Tarek Khalil [a,*], Jean Richert [b]

[a] Department of Physics, Faculty of Sciences(V), Lebanese University, Nabatieh, Lebanon
[b] Institut de Physique, Université de Strasbourg, 3, rue de l'Université, 67084 Strasbourg Cedex, France

ARTICLE INFO

Article history:
Received 18 February 2014
Received in revised form
10 September 2014
Accepted 18 September 2014
Available online 28 September 2014

PACS:
03.65.−w
03.65.Yz
03.67.Hk

Keywords:
Excitation of time-dependent quantum systems
Application of time–energy uncertainty relations
Quantum speed limit time
Extrema in the Margolus-Levitin expression

ABSTRACT

The conditions under which time–energy uncertainty relations derived by Deffner and Lutz [10] for time-dependent quantum systems minimize the time necessary to excite such systems from their ground state to excited states are examined. The generalized Margolus–Levitin and Mandelstam–Tamm inequalities are worked out for specific fermionic and bosonic systems.

1. Introduction

Time is an essential concept for the realization of fast electronic devices. As an example it may be useful to control and minimize the time needed by a two-level quantum system to jump from its ground state to an excited state. This time is constrained by Heisenberg's time–energy uncertainty relation and bounded by a lower amount known as a quantum speed limit (QSL) time.

The optimization of the time duration of quantum jumps for stationary systems has been the object of numerous theoretical studies starting with Mandelstam and Tamm [1] and pursued up to present time in order to attain the sharper lower bound [2–6].

A vast literature has been devoted to various aspects of the problem of time dependent systems [7–10]. Anandan and Aharonov have used alternative geometric derivations to obtain expressions for the Fubini-Study metric where the shortest possible distance between orthogonal states, which is along a geodesic, leads to get implicit bounds for the time of evolution of a quantum system [7,8]. Other authors have used differential geometric methods to get sharper uncertainty relations for mixed states that can be optimized in the case for fully distinguishable states. Moreover they characterize the Hamiltonians that optimize the evolution time for finite-level quantum systems [9]. Recently for arbitrary quantum unitary processes Deffner and Lutz [10] extended the Mandelstam–Tamm (MT) and the Margolus–Levitin (ML) inequalities to time dependent systems which are either intrinsically time-dependent or driven by an external time-dependent perturbation. To this end, they derive the upper bounds for the Bures angle (Bures length) or, rather, for Fisher information [6]. At this stage we optimize the time in the extended ML formulation with respect to a maximum value. Besides, this principle of optimization, that fixes the minimum time, is also applicable for all the estimates presenting an extremum. Another groups have explored the MT bound, which is geometric in nature, to attain the minimum time of evolution in the context of the time-optimal control (time-OC) problem [12–15]. More specifically, this method seems to be useful to minimize the decoherence for a system [16]. Further works have been developed for both unitary and non-unitary processes [11,17,18].

In practice it is of interest to test the time needed by a quantum system to be driven from its ground state into excited states. In the present work the inequalities derived in [10] have been used and

* Corresponding author.
E-mail addresses: tkhalil@ul.edu.lb (T. Khalil), j.mc.richert@gmail.com (J. Richert).

http://dx.doi.org/10.1016/j.physb.2014.09.027
0921-4526/

worked out in order to determine the time needed to generate quantum transitions. In the first part of the present work we examine the general conditions under which the ML inequality is optimized i.e. comes closest to an equality for specific values of the time interval over which the system evolves. In the second part the ML and MT relations are applied to fermionic and bosonic systems.

In Section 2 the inequalities are explicitly formulated and the conditions under which the inequalities can be optimized in the ML case are presented in Section 3. Section 4.1 is devoted to a quantitative study of the MT and ML expressions by applying them to a fermionic $1d$ quantum chain. In Section 4.2 the ML inequality is applied to a simple bosonic system coupled to an external time-dependent perturbation which can act in a weak or a strong coupling regime.

2. Time–energy inequalities

The general structure of the time–energy inequality can be written as [10]

$$\tau \geq R(\tau) \tag{1}$$

where $R(\tau)=\hbar C_{0_\tau}/\Delta E(\tau)$, $C_{0_\tau}=\arccos \Omega_{0_\tau}$, $\Omega_{0_\tau}=|\langle\Psi(0)|\Psi(\tau)\rangle|$ is the overlap between the wave function at time $t=0$ and $t=\tau$, and $\Delta E(\tau)$ characterizes the energy difference acquired by the system between the initial and the final time. In the following $\hbar=1$

In the Mandelstam–Tamm formulation the energy denominator $\Delta E(\tau)$ of Eq. (1) is given in terms of the variance of the energy:

$$\Delta E(\tau)=1/\tau\int_0^\tau dt\frac{[\langle H(t)^2\rangle-\langle H(t)\rangle^2]^{1/2}}{\langle\Psi(t)|\Psi(t)\rangle} \tag{2}$$

with

$$\langle H(t)^2\rangle=\langle\Psi(t)|H(t)^2|\Psi(t)\rangle \tag{3}$$

In the Margolus–Levitin formulation $\Delta E(\tau)=\langle E(\tau)\rangle-E(0)$ where $E(0)$ is the energy at $t=0$ and $\langle E(\tau)\rangle=1/\tau\int E(t)\,dt$ is the average energy of the system over a time interval $[0,\tau]$.

3. Extrema in the Margolus–Levitin expression

In the case of a time-independent system the realization of Eq. (1) as a strict equality can be of great practical interest since it leads to the determination of the minimum time needed by the system starting from its ground state to excited states. This question has been successfully answered in [5,7].

The question may also be raised in the case where the Hamiltonian dynamics are time-dependent when it is of interest to find out the minimal time interval $[0,\tau]$ needed in order to realize the equality. It comes out that an analytic solution has not been found yet. A priori a less ambitious empirical answer might correspond to $C_{0_\tau}=0$ and a minimum of the energy denominator $\Delta E(\tau)$.

A rigorous answer to the optimization of the inequality towards an equality consists in a determination of the maximum of $R(\tau)$ which brings the r.h.s. of the expression closest or equal to τ.

The first derivative of this quantity with respect to τ leads to

$$dR(\tau)/d\tau=\frac{\hbar[\Delta E(\tau)dC_{0_\tau}/d\tau-C_{0_\tau}d\Delta E(\tau)/d\tau]}{\Delta^2E(\tau)} \tag{4}$$

An extremum is reached if $dR(\tau)/d\tau=0$ which leads to

$$\frac{dC_{0_\tau}/d\tau}{C_{0_\tau}}=\frac{d\Delta E(\tau)/d\tau}{\Delta E(\tau)} \tag{5}$$

This extremum is a maximum if, for the corresponding value of τ,

$$\Delta E(\tau)d^2C_{0_\tau}/d\tau^2-C_{0_\tau}d^2\Delta E(\tau)/d\tau^2<0 \tag{6}$$

A consequence of Eq. (5) can be observed if C_{0_τ} is maximized. This corresponds to $\Omega_{0_\tau}=|\langle\Psi(0)|\Psi(\tau)\rangle|=0$, the vectors $|\Psi(\tau)\rangle$ and $|\Psi(0)\rangle$ are orthogonal to each other. Then $dC_{0_\tau}/d\tau=0$ and induces $d\Delta E(\tau)/d\tau=0$ if $\Delta E(\tau)\neq 0$. But, in the ML formulation

$$d\Delta E(\tau)/d\tau=\frac{E(\tau)}{\tau}-\frac{\langle E(\tau)\rangle}{\tau}=0 \tag{7}$$

which leads back to the expression of $\Delta E(\tau)$ if $E(0)=0$. Hence the stationarity of $\Delta E(\tau)$ is correlated with the orthogonality of the vectors $|\Psi(\tau)\rangle$ and $|\Psi(0)\rangle$. The stationary point τ can be an inflexion point or an extremum. Then $C_{0_\tau}=\pi/2\ \mathrm{mod}(k\pi)$ and

$$\tau\geq\frac{\hbar\pi/2}{\Delta E(\tau)} \tag{8}$$

Whether or not this limit can be reached and $\geq$ replaced by a strict equality depends on the system.

4. Models and applications

4.1. Fermionic 1d chain

In the first step the time–energy inequality is applied to the time-dependent $1d$ chain with even periodic boundary conditions already introduced in [19,20]

$$H_0=J/2(1+\gamma)\sum_{(i)}\sigma_i^x\sigma_{i+1}^x+J/2(1-\gamma)\sum_{(i)}\sigma_i^y\sigma_{i+1}^y-h_0\sum_{(i)}\sigma_i^z \tag{9}$$

where σ_i^x is the x component of the Pauli matrix and similarly for the y and z components. The system is integrable and the wave function is given as a product of single particle wave functions with corresponding energies [21]. The time dependence is generated by a local excitation of the last spin by an external magnetic field

$$H_1^{(N)}(t)=h_1\,\exp(-t/\tau_H)S_N^z \tag{10}$$

with $S_N^z=\sigma_N^z/2$ [21].

The wave function of the system is obtained perturbatively, up to second order in the interaction which works as a perturbation and leads to the expression of the overlap

$$\Omega_{0_\tau}=|\langle\Psi(0)|(1+U^{(1)}(0,\tau)+U^{(2)}(0,\tau))|\Psi(0)\rangle| \tag{11}$$

where $|\Psi(0)\rangle$ is the wave function at $t=0$, and $U^{(1)}(0,\tau)$ and $U^{(2)}(0,\tau)$ are respectively the first and the second order contribution to the evolution operator $U(0,\tau)=\exp[-i(H_0\tau+\int_0^\tau dtH_1^{(N)}(t))]$. A justification for neglecting higher order contributions used in the numerical application is given in the Appendix.

The energy of the system over a time interval $[0,\tau]$ reads

$$E(\tau)=1/\tau\int_0^\tau\langle\Psi(t)|H(t)|\Psi(t)\rangle/\langle\Psi(t)|\Psi(t)\rangle-E(0) \tag{12}$$

where $|\Psi(t)\rangle$ is the perturbed wave function evaluated up to order 2 and $E(0)$ the initial energy of the system.

4.1.1. General considerations concerning the application of the model

The present external magnetic field on a unique spin state of the chain has been explicitly chosen to produce a weak effect on the chain in order to allow for a perturbative treatment. Under these conditions and for fixed $J=1$ it comes out that the wave function overlaps are not very sensitive to the strength of the magnetic fields, neither h_0 nor h_1 as long as these quantities stay in the range of unity. The same is true in the case of variations of the asymmetry parameter γ which are fixed in the interval $[0,1]$. In all applications the length of the chain is $N=100$.

4.1.2. Application of the Mandelstam–Tamm expression

Typical results concerning the r.h.s. of the MT expression are shown in Fig. 1 as a function of γ with $\tau=100$, $h_0=1$, $h_1=1$. The results show that $R(\tau)$ is very small when compared with τ.