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An inverse problem of parameter estimation for time-fractional heat conduction in a composite medium using carbon–carbon experimental data

Qiao Zhuang, Bo Yu, Xiaoyun Jiang*

School of Mathematics, Shandong University, Jinan 250100, PR China

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ABSTRACT

In this paper, a time-fractional heat conduction problem is mathematically proposed for an experimental heat conduction process in a 3-layer composite medium. A numerical solution to the direct problem is obtained with finite difference method. In regard to the inverse problem, the optimal order of Caputo fractional derivative is estimated with Levenberg–Marquardt method. Comparing with the carbon–carbon experimental data, the results show that the time-fractional heat conduction model provides an effective and accurate simulation of the experimental data. The rationality of the proposed time-fractional model and validity of Levenberg–Marquardt method in solving the time-fractional inverse heat conduction problem are also manifested according to the results. By conducting the sensitivity analysis, the feasibility of the parameter estimation is further discussed.

1. Introduction

Fractional calculus, which is widely applied to various fields [1–3] such as biology, physics and finance, has become a booming branch of applied mathematics. Among its applications, anomalous diffusion or heat conduction [4–16] is considered to be one of the most popular directions because effective and concise mathematical simulations of these processes in heterogenous or disordered mediums are widely provided. In latest studies, time-fractional heat conduction models have been used to describe heat conduction processes in one-dimensional composite mediums. Jiang and Chen [8] provided an analytical and a numerical solution to a time-fractional heat conduction problem in a finite one-dimensional composite medium. Povstenko [11] delivered a time-fractional heat conduction model in an infinite one-dimensional composite medium of two layers.

Date back to 1950s, publications of inverse heat conduction problems (IHCP) began to appear [17]. The aim of IHCP is to estimate parameters, e.g., thermal conductivity, volumetric heat capacity, heat flux, etc. from the interior temperature distribution [18,19]. Then plenty of methods have been developed for IHCP and many of them, including function specification [18,20], Newton–Raphson [21], Levenberg–Marquardt (LM) [22,23], conjugate gradient [23–25], function spline [21,26], recursive least squared method [27] etc. are under the principle of the least squares [18,19,22–30] or its regularized version [18,23,25,28,29]. In recent years, fractional calculus started to be brought into the IHCP. Murio [4] set up a regularized space marching scheme to handle a time-fractional inverse heat conduction problem (TFIHCP) with the reconstruction of boundary conditions. Ghazizadeh et al. [6], estimating the relaxation parameter and the order of fractional derivative simultaneously, have found that a time-fractional single-phase-lag heat conduction constitutive model is equivalent to a classical dual-phase-lag one. However, a TFIHCP in a composite medium and its relevant analysis with experimental data have not been discussed so far, which motivates us to launch the present study.

This paper is arranged in the following layout: In Section 2, a time-fractional heat conduction model in a 3-layer composite medium is built. The numerical solution to the direct problem with finite difference method is provided in Section 3. In Section 4, Levenberg–Marquardt method has been employed to estimate the optimal order of Caputo fractional derivative. The results are obtained and analyzed with the experimental data in Section 5. At last, the conclusion of this paper has been drawn in Section 6.

2. Mathematical model for time-fractional heat conduction problem in composite medium

In this section, we consider an anomalous heat conduction process occurring in a 3-layer composite medium. The constituent

* Corresponding author.
E-mail addresses: zhqsdu@163.com (Q. Zhuang), wqjxyf@sdu.edu.cn (X. Jiang).

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layers are in good thermal contact with the inner surface of the first layer effected by the constant heat flux and the outer surface of the third layer adiabatic from the outside environment. Based on the fractional Fick's law, a mathematical model for this anomalous heat conduction can be built as follows [8,11–16]:

$$\frac{\partial^\gamma T_i(x,t)}{\partial t^\gamma}=D_{\gamma_i}\frac{\partial^2 T_i(x,t)}{\partial x^2},\quad i=1,2,3;\ L_{i-1}<x<L_i,\ 0<\gamma\le 1,\ t>0, \tag{1}$$

$$-k_1\frac{\partial_{RL}^{1-\gamma}}{\partial t^{1-\gamma}}\frac{\partial T_1(x,t)}{\partial x}=q_0,\quad x=0,\ t>0, \tag{2}$$

$$k_i\frac{\partial_{RL}^{1-\gamma}}{\partial t^{1-\gamma}}\frac{\partial T_i(x,t)}{\partial x}=k_{i+1}\frac{\partial_{RL}^{1-\gamma}}{\partial t^{1-\gamma}}\frac{\partial T_{i+1}(x,t)}{\partial x},\quad i=1,2;\ x=L_i,\ t>0, \tag{3}$$

$$T_i(x,t)=T_{i+1}(x,t),\quad i=1,2;\ x=L_i,\ t>0, \tag{4}$$

$$\frac{\partial T_3(x,t)}{\partial x}=0,\quad x=L_3,\ t>0, \tag{5}$$

$$T_i(x,t)=T_0,\quad i=1,2,3;\ t=0, \tag{6}$$

where $T_i(x,t)$ is the temperature distribution of each layer, T_0 is the initial temperature of each layer, q_0 is the value of constant heat flux, k_i is the thermal conductivity of the ith layer, D_{γ_i} represents the fractional heat conduction coefficient of the ith layer (the physical dimension of D_{γ_i} is $[D_{\gamma_i}]=\mathrm{m}^2\mathrm{s}^{-\gamma}$ [31]), $\partial^\gamma/\partial t^\gamma$ is the Caputo time-fractional derivative defined as [32]

$$\frac{\partial^\gamma f(t)}{\partial t^\gamma}=\begin{cases}\dfrac{1}{\Gamma(n-\gamma)}\int_0^t\frac{f^{(n)}(\tau)}{(t-\tau)^{\gamma-n+1}}\,\mathrm{d}\tau, & n-1<\gamma<n,\\ f^{(n)}(t), & \gamma=n,\end{cases} \tag{7}$$

and $\partial_{RL}^{1-\gamma}/\partial t^{1-\gamma}$ is the Riemann–Liouville time-fractional derivative defined as [32]:

$$\frac{\partial_{RL}^{\beta} f(t)}{\partial t^\beta}=\frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_0^t\frac{f(\tau)}{(t-\tau)^{\beta-n+1}}\,\mathrm{d}\tau,\quad n-1\le\beta<n, \tag{8}$$

where n is a positive integer.

Apply Riemann–Liouville integral $I^{1-\gamma}$ with respect to t to the both sides of Eqs. (2) and (3) and neglect the initial heat flux, we have:

$$-k_1\frac{\partial T_1(x,t)}{\partial x}=\frac{q_0 t^{1-\gamma}}{\Gamma(2-\gamma)},\quad x=0,\ t>0, \tag{9}$$

$$k_i\frac{\partial T_i(x,t)}{\partial x}=k_{i+1}\frac{\partial T_{i+1}(x,t)}{\partial x},\quad i=1,2;\ x=L_i,\ t>0, \tag{10}$$

where Riemann–Liouville integral is defined as [32] :

$$I^\beta f(t)=\frac{1}{\Gamma(\beta)}\int_0^t(t-\tau)^{\beta-1}f(\tau)\,\mathrm{d}\tau,\quad \beta>0. \tag{11}$$

Therefore, the fractional heat conduction model is considered in the form of Eqs. (1), (9), (10) and Eqs. (4)–(6).

3. Numerical solution to the direct problem

The objective of the direct problem is to solve the temperature distribution $T_i(x,t)$ from Eqs. (1), (9), (10) and Eqs. (4)–(6), where k_i, D_{γ_i}, L_i, q_0, T_0 are known. A numerical solution to the direct problem can be deduced based on some previous work of other researchers [8,10] with finite difference method. To obtain the discrete form, the homogenous segmentation of space is introduced as: $0=L_0=x_{1,0}<x_{1,1}<\cdots<x_{1,N}=L_1=x_{2,0}<x_{2,1}<\cdots<x_{2,N}=L_2=x_{3,0}<x_{3,1}<\ldots<x_{3,N}=L_3$, with $\Delta x_i=x_{i,j}-x_{i,j-1}$. And the homogenous segmentation of time can be introduced as: $0=t_0<t_1<t_2\ldots<t_n=t$, with $\Delta t=t_l-t_{l-1}$.

Denote $T_i(x_{i,j},t_k)=T_{i,j}^k$ $(i=1,2,3;\ j=1,2,\ldots,N;\ k=0,1,\ldots,n)$, the governing equation (1) has the discrete form [8,10]:

$$\frac{1}{(\Delta t)^\gamma\Gamma(2-\gamma)}\sum_{l=0}^{k}(T_{i,j}^{k-l+1}-T_{i,j}^{k-l})b_l=D_{\gamma_i}\frac{T_{i,j+1}^{k+1}-2T_{i,j}^{k+1}+T_{i,j-1}^{k+1}}{\Delta x_i^2}, \tag{12}$$

where $b_l=(l+1)^{1-\gamma}-l^{1-\gamma}$, then the discrete boundary conditions corresponding to Eqs. (9) and (10) and Eqs. (4) and (5) can be written as follows:

$$-k_1\frac{T_{1,1}^{k+1}-T_{1,0}^{k+1}}{\Delta x_1}=\frac{q_0[(k+1)\Delta t]^{1-\gamma}}{\Gamma(2-\gamma)}, \tag{13}$$

$$k_i\frac{T_{i,N}^{k+1}-T_{i,N-1}^{k+1}}{\Delta x_i}=k_{i+1}\frac{T_{(i+1),1}^{k+1}-T_{(i+1),0}^{k+1}}{\Delta x_{i+1}},\quad i=1,2, \tag{14}$$

$$T_{i,N}^{k+1}=T_{(i+1),0}^{k+1},\quad i=1,2, \tag{15}$$

$$\frac{T_{3,N}^{k+1}-T_{3,N-1}^{k+1}}{\Delta x_3}=0, \tag{16}$$

and the discrete initial condition can be written as

$$T_{i,j}^0=T_0. \tag{17}$$

As a result, an implicit finite difference scheme for solving $T_{i,j}^k$ is constructed with Eqs. (12)–(17). Follow Ref. [8], the scheme can be proved to be uniquely solvable with respect to $T_{i,j}^k$. Therefore, $T_i(x,t)=T_i(x,t,\gamma)$ is able to be numerically calculated for a given order of Caputo derivative γ.

4. Estimation of optimal order of Caputo fractional derivative with Levenberg–Marquardt method

The estimation of optimal order of Caputo fractional derivative falls into the category of solving time-fractional inverse heat conduction problem (TFIHCP). For the present TFIHCP, the objective is to seek the optimal order of Caputo derivative γ to minimize the following least squares norm:

$$S(\gamma)=(\boldsymbol{Y}-\boldsymbol{T})^{\mathrm{T}}(\boldsymbol{Y}-\boldsymbol{T}), \tag{18}$$

where $\boldsymbol{Y}$ is a vector of measured temperatures (experimental data), $\boldsymbol{T}$ is a vector of estimated temperatures. The symbol $\boldsymbol{Y}$ and $\boldsymbol{T}$ can be denoted as $\boldsymbol{Y}=(Y_{l_1},Y_{l_2},\ldots,Y_{l_M})^{\mathrm{T}}$, $\boldsymbol{T}=(T_{l_1}(\gamma),T_{l_2}(\gamma),\ldots,T_{l_M}(\gamma))^{\mathrm{T}}$, where Y_{l_j} is the measured temperature adopted from the location $x=x_{i_j}$ at time node $t=t_{l_j}$, $T_{l_j}(\gamma)=T_i(x_{i_j},t_{l_j},\gamma)$,with i = 1, 2, 3 and $j=1,2,\ldots,M$ herein. Therefore, the least squares norm Eq. (18) can be written in the scalar form as

$$S(\gamma)=\sum_{j=1}^{M}(Y_{l_j}-T_{l_j}(\gamma))^2, \tag{19}$$

as a kind of inverse problem, TFIHCP is ill-posed [23,29]. Consequently, small errors of measured data may lead to large errors in parameter estimations [29]. Thus a zeroth order Tikhonov regularization is usually added to the least squares norm Eq. (18) or Eq. (19) to avoid the ill-posedness. However, the regularization term can be neglected when the number of parameter being estimated is few [23], then solving TFIHCP based on minimizing the ordinary least squares norm Eq. (18) or Eq. (19) is reasonable in the current study where the order of Caputo derivative is the only parameter to be estimated.

Levenberg–Marquardt (LM) method, which is a combination of Newton method that converges fast with the requirement of a good initial guess and steepest decent method without the necessity of a good initial guess [23], is a widely used technique for solving nonlinear least squares problem. The LM iterative scheme with respect to the order of Caputo derivative being