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Generalized Prandtl–Ishlinskii operators arising from homogenization and dimension reduction ☆

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ABSTRACT

We consider rate-independent evolutionary systems over a physical domain Ω that are governed by simple hysteresis operators at each material point. For multiscale systems where ε denotes the ratio between the microscopic and the macroscopic length scale, we show that in the limit $\varepsilon \to 0$ we are led to systems where the hysteresis operators at each macroscopic point is a *generalized Prandtl–Ishlinskii operator*.

1. Introduction

We are interested in the generation of complicated hysteresis in the process of taking multiscale limits where the underlying problem on the small scale is described by simple hysteresis loops. For instance, we will show that homogenization of a problem with classical play operators, which do not have interior hysteresis loops, on the small scale will give rise to a homogenized macroscopic problem on the larger scale that has a complicated hysteresis operator of Prandtl–Ishlinskii type, which displays interior loops.

Our theory is based on the energetic formulation of rate-independent systems (RIS) $(\mathcal{Q},\mathcal{E},\mathcal{R})$ where the hysteresis is described by a differential inclusion for the state variable $q:[0,T]\to\mathcal{Q}$, namely

$$0 \in \partial\mathcal{R}(\dot{q})+\mathrm{D}\mathcal{E}(t,q). \tag{1.1}$$

Here $\mathcal{E}$ is the energy potential, and the dissipation potential $\mathcal{R}$ is nonnegative, convex and homogeneous of degree 1, which leads to rate independency. The set $K^* := \partial\mathcal{R}(0)\subset\mathcal{Q}^*$ is called the play domain and its boundary is called the yields surface. In the case that the energy is quadratic in q, viz.

$$\mathcal{E}(t,q)=\tfrac{1}{2}\langle Aq,q\rangle-\langle \ell(t),q\rangle,$$

we call the solution operator for Eq. (1.1) the *play operator* (cf. Refs. [5,14]) associated to A and K^* and write

$$q(t)=\mathfrak{P}_{A,K^*}[q(0),\ell](t)$$

for the output $q\in \mathrm{W}^{1,1}([0,T];\mathcal{Q})$, where $q(0)$ and $\ell\in\mathrm{W}^{1,1}([0,T];\mathcal{Q}^*)$ are the inputs. Applications include elastoplasticity, isothermal shape-memory materials, piezo-electric materials, or micromagnetism. We refer to the surveys [31,5,1,18,19] for further details on applications.

In Section 2 we recall the general theory of convergence of RIS $(\mathcal{Q},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ where ε is a small parameter tending to 0. As a special case of the abstract theory of Γ-convergence of energetic solutions derived in Ref. [24] we present a fairly general convergence theory for play operators. Under the assumption that $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ converge in the sense of Mosco to $\mathcal{E}_0$ and $\mathcal{R}_0$, respectively, and that $\mathcal{R}_\varepsilon$ continuously converges to $\mathcal{R}_0$ we have the following statement (cf. Ref. [17]): If

$$q^\varepsilon(0)\to q^0(0) \quad\text{and}\quad \mathcal{E}_\varepsilon(0,q^\varepsilon(0))\to\mathcal{E}_0(0,q^0(0))$$

then the solutions q^ε satisfy $q^\varepsilon(t)\to q^0(t)$ for all $t\in[0,T]$. The latter statement is the definition of the Γ-convergence of the RIS $(\mathcal{Q},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ to $(\mathcal{Q},\mathcal{E}_0,\mathcal{R}_0)$. Application of this theory will be given in homogenization and in dimension reductions in Sections 4 and 5.

The Γ-convergence theory shows that the set of abstract play operators is closed under Γ-convergence for RIS. In this work we want to highlight that in such limit processes the class of simple hysteresis operators is not closed. In particular, we want to show that in limits for multiscale systems we can generate complex hysteresis operators in the large-scale system, when starting with simple hysteresis operators for the small-scale system. These hysteresis operators are obtained as *symmetric B-contraction of a*

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symmetric play operator, namely

$$\mathcal{P}^B_{A,K^*}[q_0,\tilde{\ell}] := B\mathfrak{P}_{A,K^*}[q_0,B^*\tilde{\ell}].$$

We call these hysteresis operators *generalized Prandtl–Ishlinskii operators* (gPI operators). See Section 3 for further details and nontrivial examples.

In Section 4 we show how these operators appear in homogenization of elastoplastic materials, where the material properties are periodically modulated on the small scale with period ε. The mathematical tools is two-scale homogenization (cf. Refs. [2,23,26,32,28,29]), where the micro-cell problem defines the gPI operator. According to Refs. [23,8,12], the case of linearized elastoplasticity, where $q=(u,z)$ with the displacement $u:\Omega\to\mathbb{R}^d$ and the internal variable $z:\Omega\to\mathbb{R}^m$, one finds different macroscopic elastoplastic models depending on the strength of the gradient regularization $\varepsilon^{2\gamma}|\nabla z|^2$. In the case $\gamma<1$ one obtains classical models with homogenized elasticity and averaged yield strength. We refer also to Ref. [7] for such a result in space dimension 1. However, for $\gamma\geq 1$ the macroscopic model can only be described in terms of a gPI operator. The occurrence of more complicated hysteresis operators for homogenized material models was also highlighted in Refs. [32,33].

In Section 5 we recall the rigorous derivation of an elastoplastic plate model from Refs. [16,17]. We show that it has a natural interpretation in terms of vector-valued gPI operators. While the case of pure bending was treated in Ref. [11], we follow the general approach of Ref. [17], where membrane and bending deformations are coupled via plastic effects.

2. Γ-Convergence **for rate-independent systems**

Here we consider general families $(\mathcal{Q},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)_{\varepsilon\in]0,1]}$ of RIS and study the convergence of the associated solutions q^ε in the limit $\varepsilon\to 0$. The aim is to establish fairly general conditions on the convergences of $(\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ to $(\mathcal{E}_0,\mathcal{R}_0)$ that guarantee that the solutions q_ε converge to the solution q of the limit system $(\mathcal{Q},\mathcal{E}_0,\mathcal{R}_0)$, which we then call the Γ-limit of the above family.

For rate-independent systems a general strategy for Γ-convergence was developed in Ref. [24], which found numerous applications in, e.g., fracture [9], homogenization [23], numerical approximation [13,10,20], and delamination [27,25]. Here we specialize this theory to the case that $\mathcal{E}_\varepsilon(t,\cdot):\mathcal{Q}\to\mathbb{R}_\infty$ is a quadratic functional, as it is the case for play operators and in linearized elastoplasticity. Thus, the abstract theory is simplified in two respects. First, the systems under consideration have unique solutions and we do not need to consider subsequences. Second, the quadratic nature of the energy allows for a simpler construction of recovery sequences by using the quadratic trick introduced in Ref. [23]. Thus, the strong compactness assumptions in Ref. [24] can be avoided.

The convergence result is formulated abstractly in terms of Γ-convergence of $\mathcal{E}_\varepsilon(t,\cdot)$ towards $\mathcal{E}_0(t,\cdot)$ and of $\mathcal{R}_\varepsilon$ to $\mathcal{R}_0$, where we use the weak and the strong topologies in the underlying separable Hilbert space $\mathcal{Q}$. It might be surprising that convergence of the functionals $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ is enough to guarantee convergence of the solutions q^ε of the subdifferential inclusion (1.1), since from the equation it seems necessary to control the convergence of the (sub-) differentials. The relevance of the functionals is seen better if we use the equivalent *energetic formulation* for RIS. Here the equivalence holds, as $\mathcal{E}_\varepsilon(t,\cdot)$ is strictly convex, see Refs. [21,18]. A function q^ε is called *energetic solution* for the RIS $(\mathcal{Q},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ if for all $t\in[0,T]$ we have the stability (S) and the energy balance (E):

(E) $\mathcal{E}_\varepsilon(t,q^\varepsilon(t))\leq\mathcal{E}_\varepsilon(t,\tilde{q})+\mathcal{R}_\varepsilon(\tilde{q}-q^\varepsilon(t))$ for all $\tilde{q}\in\mathcal{Q}$;

(S) $\mathcal{E}_\varepsilon(t,q^\varepsilon(t))+\mathrm{Diss}_{\mathcal{R}_\varepsilon}(q^\varepsilon,[0,t])=\mathcal{E}_\varepsilon(0,q^\varepsilon(0))+\int_0^t\partial_s\mathcal{E}_\varepsilon(s,q^\varepsilon(s))\,\mathrm{d}s.$

The dissipation $\mathrm{Diss}_{\mathcal{R}}(q,[r,s])$ is defined via

$$\mathrm{Diss}_{\mathcal{R}}(q,[r,s]) := \sup\sum_{j=1}^N\mathcal{R}(q(t_j)-q(t_{j-1})),$$

where the supremum is taken over all $N\in\mathbb{N}$ and all partitions $r<t_0<t_1<\cdots<t_N<s$. Note that the dissipation is defined along any curve $q:[0,T]\to\mathcal{Q}$ without any assumptions on continuity or differentiability. For absolutely continuous functions we have

$$\mathrm{Diss}_{\mathcal{R}}(q,[r,s])=\int_r^s\mathcal{R}(\dot{q}(t))\,\mathrm{d}t.$$

We recall that the energetic formulation via (S) and (E) is totally equivalent to the subdifferential inclusion for play operators, where $\mathcal{E}_\varepsilon(t,\cdot)$ is uniformly convex. Its importance is that it is totally derivative free. We neither need derivatives of the solution $q^\varepsilon:[0,T]\to\mathcal{Q}$ nor of the functionals $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$. Thus, it is ideally suited for limiting processes in the variational sense, where the convergence of functionals is studied, see Refs. [3,6]. We use the notions of *Mosco convergence* and *continuous convergence* for functionals $\mathcal{I}_n$. The first is written $\mathcal{I}_n\xrightarrow{M}\mathcal{I}$ and defined via (i) and (ii):

(i) Liminf estimate:

$$q_n\rightharpoonup q \quad\Longrightarrow\quad \mathcal{I}(q)\leq\liminf_{n\to\infty}\mathcal{I}_n(q_n),$$

(ii) Limsup estimate ($\hat{=}$ $\exists$ recovery sequences)

$$\forall\,\hat{q}\in\mathcal{Q}\ \exists\,(\hat{q}_n)_n:\quad \hat{q}_n\to\hat{q}\quad\text{and } \mathcal{I}(\hat{q})\geq\limsup_{n\to\infty}\mathcal{I}_n(\hat{q}_n).$$

The continuous convergence (with respect to the norm topology) is written as $\mathcal{I}_n\overset{c}{\rightsquigarrow}\mathcal{I}$ and defined via

$$\mathcal{I}_n\overset{c}{\rightsquigarrow}\mathcal{I}\ \Leftrightarrow\ (q_n\to q\ \Rightarrow\ \mathcal{I}_n(q_n)\to\mathcal{I}(q)).$$

Our precise assumptions on the family $(\mathcal{Q},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)_{\varepsilon\in[0,1]}$ are the following. Note that often the limit functionals $\mathcal{E}_0$ and $\mathcal{R}_0$ are included in the assumptions via $\varepsilon=0$. The assumptions (2.1a)–(2.1c) provide some uniform a priori estimates, while Eqs. (2.1d) and (2.1e) are the main convergence assumptions:

$$\mathcal{E}_\varepsilon(t,q)=\mathcal{B}_\varepsilon(q)-\langle\ell_\varepsilon(t),q\rangle \text{ where } \mathcal{B}_\varepsilon \text{ is quadratic, wlsc and } \ell_\varepsilon\in\mathrm{C}^1([0,T];\mathcal{Q}^*); \tag{2.1a}$$

$$\mathcal{R}_\varepsilon:\mathcal{Q}\to[0,\infty] \text{ is 1−homogeneous, wlsc, and convex;} \tag{2.1b}$$

$$\exists\beta,C>0\quad\forall(t,q)\in[0,T]\times\mathcal{Q}\ \forall\varepsilon\in[0,1]: \quad \mathcal{B}_\varepsilon(q)\geq\frac{\beta}{2}\|q\|^2,\quad \|\ell_\varepsilon(t)\|_{\mathcal{Q}^*}+\|\dot{\ell}_\varepsilon(t)\|_{\mathcal{Q}^*}\leq C; \tag{2.1c}$$

$$\mathcal{B}_\varepsilon\xrightarrow{M}\mathcal{B}_0\quad\text{and }\forall t:\ \ell_\varepsilon(t)\to\ell_0(t)\text{in}\mathcal{Q}^*; \tag{2.1d}$$

$$\mathcal{R}_\varepsilon\overset{c}{\rightsquigarrow}\mathcal{R}_0\quad\text{and }\mathcal{R}_\varepsilon\xrightarrow{M}\mathcal{R}_0. \tag{2.1e}$$

In the last condition "$\overset{c}{\rightsquigarrow}$" implies that every strongly converging sequence is a recovery sequence. The additional condition "$\xrightarrow{M}$" is needed in order to guarantee $\mathcal{R}_0(q_0)\leq\liminf_{\varepsilon\to0}\mathcal{R}_\varepsilon(q_\varepsilon)$ whenever $q_\varepsilon\rightharpoonup q_0$. Note that we only ask for continuous convergence in the norm topology, which is in contrast to Refs. [13,24,20], where the more restrictive continuous convergence in the weak topology is used. Thus, one can follow Ref. [23] and exploit the quadratic structure (2.1a) of $\mathcal{E}_\varepsilon$ for the construction of *mutual recovery sequences*.