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A generalization of the fundamental theorem of Brown for fine ferromagnetic particles

G. Di Fratta [a,*], C. Serpico [b,**], M. d'Aquino [a,**]

[a] Department of Technology, University of Napoli "Parthenope", I-80143 Napoli, Italy
[b] Department of Engineering, University of Naples Federico II, via Claudio, 21 I-80125 Naples, Italy

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ABSTRACT

In this paper we extend the Brown's fundamental theorem on fine ferromagnetic particles to the case of a general ellipsoid. By means of Poincaré inequality for the Sobolev space $H^1(\Omega,\mathbb{R}^3)$, and some properties of the induced magnetic field operator, it is rigorously proven that for an ellipsoidal particle, with diameter d, there exists a critical size (diameter) d_c such that for $d < d_c$ the uniform magnetization states are the only global minimizers of the Gibbs–Landau free energy functional $\mathcal{G}_{\mathcal{L}}$. A lower bound for d_c is then given in terms of the demagnetizing factors.

1. Introduction

Theoretical discussions of the coercivity of magnetic materials make considerable use of the following idea [1]: "*whereas a ferromagnetic material in bulk* (*in zero applied field*) *possesses a domain structure*, *the same material in the form of a sufficiently fine particle is uniformly magnetized to* (*very near*) *the saturation value*, *or in other words consists of a single domain*".

But as Brown points out in Ref. [1]: "*the idea as thus expresses*, *scarcely is to be called a theorem*, *for it is not a proved proposition nor a strictly true one*".

The first rigorous formulation of this idea is due to Brown himself who, in his fundamental paper [1] rigorously proved for spherical particles what is known as *Brown's fundamental theorem of the theory of fine ferromagnetic particles*.

This fundamental theorem states the existence of a *critical radius* r_c of the spherical particle such that for $r < r_c$ and zero applied field the state of lowest free energy (the ground state) is one of uniform magnetization.

The physical importance of Brown's fundamental theorem is that it formally explains, although in the case of spherical particles, the high coercivity that fine particles materials have, compared with the same material in bulk [1].

In fact, if the particles are fine enough to be single domain, and magnetic interactions between particles have a negligible effect, each individual particle can reverse its magnetization only by rigid rotation of the magnetization vector of the particle as a whole, a process requiring a large reversed field (rather than by domain wall displacement, which is the predominant process in bulk materials at small fields) [1].

The main limitation of the theorem is that it is applicable to spherical particles whereas, real particles are most of the time elongated [2]. Motivated by this, Aharoni [2], by using the same mathematical reasoning as Brown, was able to extend the Fundamental Theorem to the case of a *prolate spheroid*.

The main objective of this paper is to extend, by means of Poincaré inequality for the Sobolev space $H^1(\Omega,\mathbb{R}^3)$ [3,4] and some properties of the magnetostatic self-energy [5–8], the fundamental theorem of Brown to the case of a *general ellipsoid*. In the sequel, it is rigorously proven that for an *ellipsoidal* particle, with diameter d, there exists a *critical size* (diameter) d_c such that for $d < d_c$ the uniform magnetization states are the only global minimizers of the micromagnetic free energy functional.

A lower bound for d_c is then given in terms of the demagnetizing tensor eigenvalues [9] (the so called *demagnetizing factors* [10]), which completely characterize the induced magnetic field inside ellipsoidal particles, thanks to Payne and Weinberger result on the best Poincaré constant [3,4].

2. Formal theory of micromagnetic equilibria

We start our discussion by recalling basic facts about micromagnetic theory. According to micromagnetics the local state of magnetization of matter is described by a vector field, the magnetization $\boldsymbol{m}$, defined over Ω which is the region occupied by the body.

* Principal corresponding author.
** Corresponding authors.
E-mail addresses: giovanni.difratta@uniparthenope.it (G. Di Fratta), serpico@unina.it (C. Serpico), daquino@uniparthenope.it (M. d'Aquino).

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The stable equilibrium states of magnetization are the *minimizers* of the so called Gibbs–Landau free energy functional associated with the magnetic body. In dimensionless form, and for zero applied field, this functional can be written as [1,9,11,12]:

$$\mathcal{G}_{\mathcal{L}}(\boldsymbol{m},\Omega) = \frac{1}{|\Omega|}\int_{\Omega}\left(\frac{\ell_{\text{ex}}^2}{2}|D\boldsymbol{m}|^2 - \frac{1}{2}\mathbf{h}_{\text{d}}[\boldsymbol{m}]\cdot\boldsymbol{m}\right)\text{d}v, \tag{1}$$

where $\boldsymbol{m}:\Omega\to\mathbf{S}^2$ is a vector field taking values on the unit sphere $\mathbf{S}^2$ of $\mathbb{R}^3$, and $|\Omega|$ denotes the volume of the region Ω, and ℓ_{ex}^2 is a positive material constant.

The constraint on the image of $\boldsymbol{m}$ is due to the following fundamental assumption of the micromagnetic theory: a ferromagnetic body well below the Curie temperature is always locally saturated. This means that the following constraint is satisfied:

$$|\boldsymbol{m}| = 1 \quad \text{a.e. in } \Omega. \tag{2}$$

Global micromagnetic minimizers correspond to vector fields which minimize the Gibbs–Landau energy functional (1) in the class of vector fields which take values on the unit sphere $\mathbf{S}^2$.

3. The magnetostatic self-energy. Mathematical properties of the dipolar magnetic field. The Brown lower bound

The energy functional $\mathcal{G}_{\mathcal{L}}$ given by Eq. (1) is the sum of two terms: the exchange energy and the Maxwellian magnetostatic self-energy (the second term).

The *magnetostatic self-energy* is the energy due to the (dipolar) magnetic field $\mathbf{h}_{\text{d}}[\boldsymbol{m}]$ generated by $\boldsymbol{m}$. From the mathematical point of view, assuming Ω to be open, bounded and with Lipschitz boundary, and denoting with $\chi_\Omega\boldsymbol{m}$ the trivial extension of the magnetization $\boldsymbol{m}$ to all the space $\mathbb{R}^3$, the induced magnetic field can be defined as the unique vector field $\mathbf{h}_{\text{d}}[\boldsymbol{m}]\in L^2(\mathbb{R}^3,\mathbb{R}^3)$ which satisfies (in the sense of distributions on $\mathbb{R}^3$) the following Maxwell's [7,9]:

$$\begin{cases} \text{div}(\mathbf{h}_{\text{d}}[\boldsymbol{m}]+\chi_\Omega\boldsymbol{m}) = 0 \\ \text{curl}\,\mathbf{h}_{\text{d}}[\boldsymbol{m}] = \mathbf{0}. \end{cases} \tag{3}$$

We recall that the operator $\mathbf{h}_{\text{d}}$ which to every $\chi_\Omega\boldsymbol{m}\in L^2(\mathbb{R}^3,\mathbb{R}^3)$ associates the unique solution $\mathbf{h}_{\text{d}}[\boldsymbol{m}]$ of the above Maxwell's equations, is a bounded, self-adjoint and negatively semidefined linear operator with $\|\mathbf{h}_{\text{d}}\|_{\text{op}} = 1$, when endowed with the $L^2(\Omega,\mathbb{R}^3)$ scalar product given by

$$(\boldsymbol{m},\boldsymbol{u})_\Omega = \int_\Omega \boldsymbol{m}\cdot\boldsymbol{u}\,\text{d}v. \tag{4}$$

Self-adjointness means that for every $\boldsymbol{m},\boldsymbol{u}\in L^2(\Omega,\mathbb{R}^3)$ the following equality holds:

$$(\mathbf{h}_{\text{d}}[\boldsymbol{m}],\boldsymbol{u})_\Omega = (\boldsymbol{m},\mathbf{h}_{\text{d}}[\boldsymbol{u}])_\Omega, \tag{5}$$

while semidefinite negativeness states that, for every $\boldsymbol{m}\in L^2(\Omega,\mathbb{R}^3)$, we have

$$-(\mathbf{h}_{\text{d}}[\boldsymbol{m}],\boldsymbol{m})_\Omega \geq 0. \tag{6}$$

Obviously the semidefinite negativeness of the induced magnetic field assures the positiveness of the Gibbs–Landau free energy functional.

Finally let us recall the following Brown lower bound to the magnetostatic self-energy [1,5,6] as reported by himself in Ref. [1].

Consider an arbitrary irrotational vector field $\mathbf{h}$ which is defined over the whole space $\mathbb{R}^3$ and is regular at infinity. Under these assumptions Brown proved that:

$$-\int_\Omega \mathbf{h}\cdot\boldsymbol{m}\,\text{d}v - \frac{1}{2}\int_{\mathbb{R}^3}|\mathbf{h}|^2\,\text{d}v \leq -\frac{1}{2}\int_\Omega \mathbf{h}_{\text{d}}[\boldsymbol{m}]\cdot\boldsymbol{m}\,\text{d}v, \tag{7}$$

the equality holding if and only if $\mathbf{h} = \mathbf{h}_{\text{d}}[\boldsymbol{m}]$.

In other terms, for every irrotational and regular at infinity vector field $\mathbf{h}:\mathbb{R}^3\to\mathbb{R}^3$, the left hand side of Eq. (7) does not exceed the magnetostatic self-energy and becomes equal to it only when $\mathbf{h}$ is everywhere equal to $\mathbf{h}_{\text{d}}[\boldsymbol{m}]$. It is worthwhile emphasizing that the vector field $\mathbf{h}$ in this inequality needs not be related in any way to $\boldsymbol{m}$ [2].

A very useful particular case of this lower bound can be obtained by letting $\mathbf{h} = \mathbf{h}_{\text{d}}[\boldsymbol{u}]$ with $\boldsymbol{u}\in L^2(\Omega,\mathbb{R}^3)$. In this way we arrive at the following form of the Brown lower bound which we state here as a lemma:

Lemma 1. *Let $\Omega\subseteq\mathbb{R}^3$ be open, bounded and with Lipchitz boundary. For every $\boldsymbol{u},\boldsymbol{m}\in L^2(\Omega,\mathbb{R}^3)$:*

$$-(\mathbf{h}_{\text{d}}[\boldsymbol{u}],\boldsymbol{m})_\Omega + \tfrac{1}{2}(\mathbf{h}_{\text{d}}[\boldsymbol{u}],\boldsymbol{u})_\Omega \leq -\tfrac{1}{2}(\mathbf{h}_{\text{d}}[\boldsymbol{m}],\boldsymbol{m})_\Omega, \tag{8}$$

with equality if and only if $\boldsymbol{u} = \boldsymbol{m}$.

4. The case of ellipsoidal geometry. Demagnetizing tensor

Since $\mathbf{h}_{\text{d}}$ is a linear operator, the restriction of $\mathbf{h}_{\text{d}}$ to the subspace $U(\Omega,\mathbb{R}^3)$ of constant in space vector fields can be identified with a second order tensor known as the *effective demagnetizing tensor* of Ω and defined by [9,10]:

$$N_{\text{eff}}[\boldsymbol{m}] = -\int_\Omega \mathbf{h}_{\text{d}}[\boldsymbol{m}]\,\text{d}v = -|\Omega|\langle\mathbf{h}_{\text{d}}[\boldsymbol{m}]\rangle_\Omega, \tag{9}$$

where $\boldsymbol{m}\in U(\Omega,\mathbb{R}^3)$ and for all $\boldsymbol{u}\in L^2(\Omega,\mathbb{R}^3)$ we have denoted with

$$\langle\boldsymbol{u}\rangle_\Omega = \frac{1}{|\Omega|}\int_\Omega \boldsymbol{u}\,\text{d}v, \tag{10}$$

the average of $\boldsymbol{u}$ over Ω. The tensor N_{eff} is known in literature as the *effective demagnetizing tensor* of Ω, where the qualifier effective is used as a reminder of the fact that N_{eff} is related to the average of $\mathbf{h}_{\text{d}}[\boldsymbol{m}]$ over Ω [9,10].

In addition to that, a well known result of potential theory, states that when Ω is an ellipsoid and $\boldsymbol{m}\in U(\Omega,\mathbb{R}^3)$ also $\mathbf{h}_{\text{d}}[\boldsymbol{m}]\in U(\Omega,\mathbb{R}^3)$; i.e. if Ω is an ellipsoid and $\boldsymbol{m}$ is constant, then $\mathbf{h}_{\text{d}}[\boldsymbol{m}]$ is also constant in Ω.

In physical terms this means that uniformly magnetized ellipsoids induce uniform magnetic fields in their interiors. In this case the effective demagnetizing tensor N_{eff} is pointwise related to $\boldsymbol{m}$ since the relation (9) becomes:

$$N_{\text{eff}}[\boldsymbol{m}] = -\mathbf{h}_{\text{d}}[\boldsymbol{m}]. \tag{11}$$

In the rest of the present paper we will indicate with N_{d} the demagnetizing tensor associated to an ellipsoidal particle Ω.

Obviously, from Eq. (6), we get that the quadratic form $Q_{\text{d}}(\boldsymbol{m}) = N_{\text{d}}[\boldsymbol{m}]\cdot\boldsymbol{m}$ is a definite positive quadratic form. We will indicate with

$$\mu^2 = \inf_{\boldsymbol{u}\in\mathbb{R}^3-\{\mathbf{0}\}} \frac{Q_{\text{d}}(\boldsymbol{u})}{|\boldsymbol{u}|^2}, \tag{12}$$

the first eigenvalue associated to this quadratic form, i.e. the *minimum demagnetizing factor* for the ellipsoid Ω. This quantity can be expressed analytically in terms of elliptic integrals [10].

It is important to stress that the eigenvalues of the quadratic form Q_{d} are shape-dependent but not size-dependent so that, when the volume $|\Omega|$ is changed by preserving the shape of the ellipsoid, μ^2 does not change.

5. The exchange energy and the Poincaré inequality. Null average micromagnetic minimizers

The *exchange energy* (the first term in Eq. (1)), energetically penalizes spatially non-uniform magnetization states: it takes