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On the momentum of mechanical plane waves

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Abstract

Momentum of a mechanical, harmonic plane wave is derived and explained as a relativistic effect arising from the presence of tension in moving elements of the medium. Neglect of the relativistic corrections leads to the paradox, which is formulated and explained. Explicit results for momentum density resulting from tension for transverse and longitudinal waves are discussed. The idea of experiments for quantitative measurements of the momentum density is presented.

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1. Introduction

Momentum density $\mathrm{d}\boldsymbol{p}/\mathrm{d}V$ is strictly related to the energy flowing per unit surface per unit time [1]:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}V} = \frac{1}{c^2}\boldsymbol{S} \tag{1}$$

where c is the speed of light, and can be obtained from the Lorentz transformation of momentum fourvector. In case of electromagnetic fields vector $\boldsymbol{S}$ coincides with the Poynting vector [2]. Although the relation between the energy flow and the momentum density (1) is strictly valid, precise recognition of the momentum density is sometimes not straightforward [3]. In the case of mechanical waves, in particular acoustic plane waves in elastic medium, the situation requires careful analysis. Indeed, in the canonical courses of physics [1,4,5] one cannot find discussion of the question of the momentum density of such a wave. In particular, to our knowledge one can hardly find an explicit derivation of the momentum density when a textbook example, masses connected by springs, which is introductory example for the concept of sound and phonons, is discussed. Theoretical considerations presented in Refs. [6–11] do not suggest any possibility of measuring of the momentum density of plane waves. Some published results for momentum density of plane waves disagree with Eq. (1).

Let us consider the textbook example of infinite chain of springs with stiffness constant k connected with point masses m [12]. The length of the springs at equilibrium is L. Each mass is numbered by n, and vibrates about its positions of equilibrium only. Position of the nth mass is $x_n(t)$. Deformation of the system $\Psi^{(n)}$ (see Fig. 1a) understood as a departure from the equilibrium, is described by

$$\Psi^{(n)} = x_n(t) - nL \tag{2}$$

Equation of motion is

$$\ddot{\Psi}^{(n)} = -k/m(2\Psi^{(n)} - \Psi^{(n-1)} - \Psi^{(n+1)}) \tag{3}$$

and a harmonic running wave is described by

$$\Psi^{(n)} = \Psi_0 \cos(nqL - \omega t + \varphi) \tag{4}$$

where q is the wave vector, φ constant phase and the dispersion relation, $\omega(q)$, is

$$\omega = 2\sqrt{k/m}\,\sin\frac{Lq}{2} \tag{5}$$

Eq. (4) describes a sinusoidal wave travelling from left to right. The system transmits energy. This can be visualized in *gedanken* experiment. Let us cut the system just before mass $n+1$ and attach a body that moves with a friction (see

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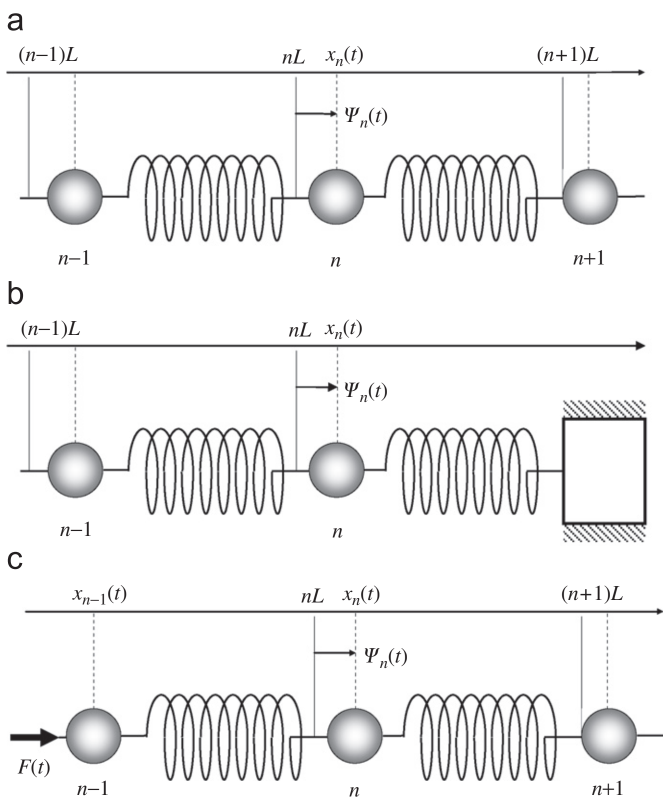

Fig. 1. (a) Part of infinite system of masses and springs, (b) transfer of energy to the receiver shown as box, (c) external force as a source of the plane wave. Solid vertical lines show equilibrium position of the balls.

Fig. 1b). We will observe dissipation of the wave energy by the body. In order to calculate the energy flow of the wave running from left to right per unit time, let us perform another *gedanken* experiment. Let us now cut the system before mass $n-1$ and apply an external force $F(t)$ such that all masses to the right will move according to Eq. (4) (see Fig. 1c). The work W per unit time of the force is

$$\frac{\mathrm{d}W}{\mathrm{d}t} = F(t)\dot{\Psi}^{(n-1)} = -k(\Psi^{(n-2)} - \Psi^{(n-1)})\dot{\Psi}^{(n-1)} \tag{6}$$

The average power transmitted to the system is the energy flow:

$$\left\langle \frac{\mathrm{d}W}{\mathrm{d}t} \right\rangle = \frac{1}{\tau}\int_{t_0}^{t_0+\tau} \frac{\mathrm{d}W}{\mathrm{d}t}\,\mathrm{d}t = \frac{1}{2}\Psi_0^2 k\omega \sin Lq \tag{7}$$

where τ is the period of the vibrations $\tau = 2\pi/\omega$. In context of the generality of Eqs. (1), (7), should result in the momentum density of the particular example of mechanical plane wave (4):

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{1}{c^2 L}\left\langle \frac{\mathrm{d}W}{\mathrm{d}t} \right\rangle = \frac{1}{c^2}\frac{\Psi_0^2 m\omega^2}{2L}\frac{\mathrm{d}\omega}{\mathrm{d}q} \tag{8}$$

The $\mathrm{d}\omega/\mathrm{d}q$ in Eq. (8) is the group velocity and can be obtained from dispersion relation (5). We do not write explicit result for $\mathrm{d}\omega/\mathrm{d}q$ in order to keep it as a group velocity.

The result (8) can be extended to three-dimensional body consisting of masses connected by springs forming simple cubic structure. For the longitudinal wave in the (1 0 0) direction we have the momentum density:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}V} = \frac{1}{c^2 L^3}\left\langle \frac{\mathrm{d}W}{\mathrm{d}t} \right\rangle \frac{\boldsymbol{q}}{q} = \frac{1}{c^2}\frac{\Psi_0^2 \rho\omega^2}{2}\frac{\mathrm{d}\omega}{\mathrm{d}q}\frac{\boldsymbol{q}}{q} \tag{9}$$

The results (8) and (9) are simple consequences of Eq. (1). The right-hand sides of Eqs. (8) and (9) are the energy density of the wave multiplied by the group velocity and divided by the square of speed of light. Let us recognize the specific form of the expressions describing the momentum density in Eqs. (8) and (9). Mechanical momentum of the nth ball is just $m\dot{\Psi}^{(n)}$. However this quantity averaged over time yields zero. Also this quantity for certain time (summed over entire system) is zero. If one treats the system as relativistic one and corrects the momentum by a relativistic factor plus a relativistic contribution coming from potential energy of the moving spring, one also gets zero momentum density. This result can serve as paradox in context of Eqs. (8) and (9). In the next paragraphs we show that the "hidden momentum" results from the tension present in the moving springs.

2. Recognition of the "hidden momentum"

Let us consider in the inertial system O an elastic medium of density ρ at the rest. Consider next an element of the same medium in which an uniaxial stress is present, see Fig. 2. The energy of the compressed element is larger by potential energy resulting from the Hooks' law. Although it is obvious, we quote that none of the forces shown in Fig. 2 is performing the work in the O system.

Let us observe the elements of the bodies from the inertial system O', which is moving with respect to the O with velocity υ. We will observe Lorentz contraction of both elements and their constant velocity movement. There is, however, remarkable difference between the energy flows in both elements. In the uncompressed element there is a flow of mass only while in the compressed one, forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are performing a work, because these forces are acting on the ends moving with velocity $-\upsilon$. So there is an additional flow of the energy through the compressed element, and according to Eq. (1) an additional momentum density should be present.

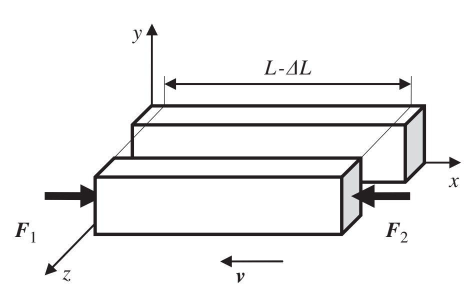

Fig. 2. Momentum density arising from stress and movement of the medium.