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Physica B 403 (2008) 301–304
PHYSICA B
www.elsevier.com/locate/physb

Hopf bifurcation in a van der Pol type oscillator with magnetic hysteresis

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Abstract

We consider models of electronic oscillators of van der Pol type assuming magnetic hysteresis in inductance element. The Preisach nonlinearity is used to model the hysteresis relation. We study local and global behaviour of the branch of cycles originating from the Hopf bifurcation point.

PACS: 75.60.−d

Keywords: Hysteresis; van der Pol oscillator; Hopf bifurcation; Preisach model

1. Introduction

A common feature of various electric circuits are oscillating LC contours. Of a particular interest are systems where the inductance element contains a ferromagnetic core, which introduces a hysteresis relation between the magnetic induction B and magnetic field H.

While this hysteresis effect can often be neglected, there are specific applications, where it is essential to include it in the model. For instance, recent studies of power transformers indicate that the existing models fail to accurately reflect the physical behaviour of the magnetic core in the presence of the phenomenon of ferroresonance. A new approach, using the Preisach operator $\mathscr{P}$ [1] as a model of the relation between B and H, was shown to provide a better agreement between the numerical results and experimental data in Ref. [2].

The purpose of this paper is to study the effect of hysteresis on periodic oscillations in a simple circuit using analytical and numerical tools. We consider van der Pol oscillators with a ferromagnetic core in the inductance element. The underlying electrical circuits consist of an LCR contour and a negative feedback loop, which can be implemented on the basis of a triode, as in the classical van der Pol circuit, or by using a tunnel diode as shown below.

The model can be written in the form

$$(z + c\mathscr{P}x)' = f(z), \quad z \in \mathbb{R}^N, \tag{1}$$

where prime denotes the time derivative, $\mathscr{P}$ is the Preisach operator, $x = \langle b, z\rangle$ with $\langle\cdot,\cdot\rangle$ denoting a scalar product in $\mathbb{R}^N$, and $b, c \in \mathbb{R}^N$. The function $f(z)$ depends on the circuit parameters, including the input–output characteristic of the triode element (or the tunnel diode). We consider a simple case with $N = 2$ and linear f, and then discuss briefly the effects of the nonlinear elements such as the cubic term in the classical van der Pol system.

2. Main example

As a prototype example, consider the system

$$(x + a\mathscr{P}x)' = y, \quad y' = -x + \lambda y, \quad a > 0, \tag{2}$$

where a is fixed, λ is a parameter, and the Preisach operator is defined by

$$(\mathscr{P}[\eta_0]x)(t) = \int_{-\infty}^{\infty}\int_{\alpha}^{\infty} \mu(\alpha,\beta)(\mathscr{R}_{\alpha,\beta}[\eta_0]x)(t)\,\mathrm{d}\beta\,\mathrm{d}\alpha.$$

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[1]This publication has emanated from research conducted with the financial support of Science Foundation Ireland and IRCSET.

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doi:10.1016/j.physb.2007.08.034

Here $\mathscr{R}_{\alpha,\beta}$ denotes a nonideal relay with thresholds α and β, and values 0 and 1; $\mu(\alpha,\beta)\geqslant 0$ is a density of a probability measure, defined on the half-plane $\alpha\leqslant\beta$ (hence $0\leqslant(\mathscr{P}[\eta_0]x)(t)\leqslant 1$ for all t); $\eta_0=\eta_0(\alpha,\beta)$ is an initial Preisach configuration on this half-plane.

To clarify the role of hysteresis, we assume that all circuit elements operate in a parameter range where their current–voltage characteristic is linear (which is fair for some range of current values), hence the right-hand part of Eq. (2) is linear too. The only nonlinearity is introduced by the Preisach operator, that is we assume the magnetic hysteresis to dominate all other nonlinear effects.

Fig. 1(a)–(c) shows the amplitude r of the cycles, both stable and unstable, plotted against the values of the parameter λ for three different measure densities of the Preisach operator and $a=1$. These plots, obtained numerically, demonstrate three common features. Firstly, there is the subcritical Hopf bifurcation at the zero equilibrium for $\lambda=0$. Secondly, the supercritical Hopf bifurcation at infinity occurs for the same $\lambda=0$. Thirdly, a continuous branch of cycles connects these two bifurcations. In Fig. 1(a), (c) the lower part of the branch corresponds to stable cycles, and the upper part corresponds to unstable cycles. Fig. 1(a) shows a fold bifurcation at $\lambda=\lambda_c$; Fig. 1(b) demonstrates three fold bifurcations and an interval of the values of λ where the equation has four co-existing cycles: two stable and two unstable.

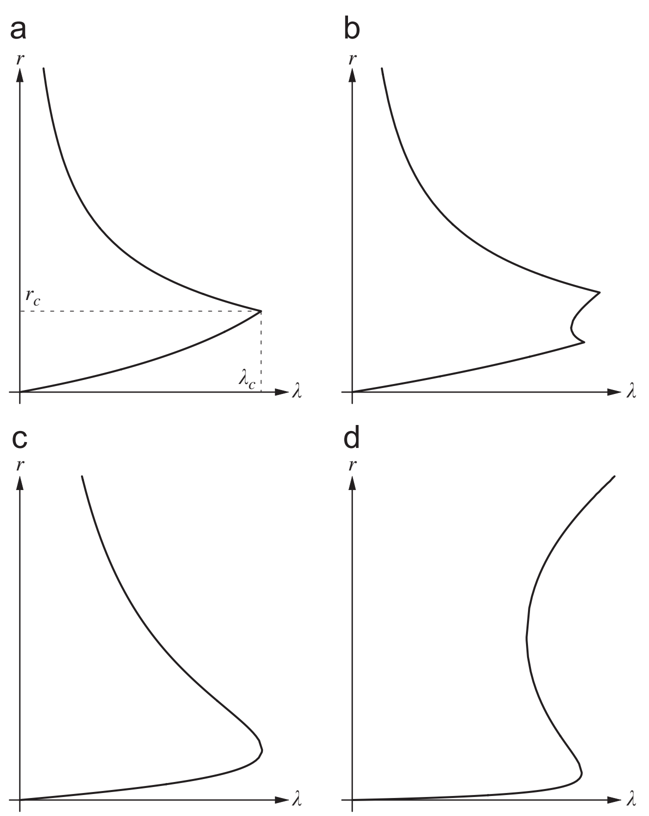

Fig. 1. (a)–(c) Amplitudes of cycles for Eq. (2) with $a=1$. (a) $\mu=0.5\chi_{\Delta(1)}$ where $\chi_{\Delta(b)}$ is a characteristic function of the triangle $\Delta(b)=\{-b\leqslant\alpha\leqslant\beta\leqslant b\}$; the cusp is associated with the jump of μ at the boundary of $\Delta(1)$, the ordinate of the cusp point is $r_c=1$. (b) $\mu=0.1(\chi_{\Delta(1)}+\chi_{\Delta(2)})$; cusp ordinates are 1 and 2. (c) $\mu=2\pi^{-1}(\alpha^2+1)^{-1}(\beta^2+1)^{-1}$ is continuous on the half plane $\alpha\leqslant\beta$. (d) cycles for Eq. (3) with $a=200$, $k_1=1$, $k_3=0.1$, $k_2=\lambda$, $\mu=0.5\chi_{\Delta(1)}$.

We will discuss in Section 4 how nonlinearities in the right-hand part deform these plots for larger values of the cycle amplitude (see Fig. 1(d)).

3. Analysis of Eq. (2)

Here we formulate three theorems about the branch of cycles of Eq. (2), which can be easily extended to a more general Eq. (1) with a linear right-hand side. More specific information on the shape of the branch of cycles of Eq. (2) (like, for example, the number of folds) and pictures similar to Fig. 1 can be obtained by numerical simulation.

Suppose that the function f in Eq. (1) depends on the scalar parameter λ, i.e. $f=f(z,\lambda):\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}^N$, and $f(0,\lambda)=0$, so that the origin is an equilibrium of Eq. (1) for all values of λ. We use the following weak definition of the Hopf bifurcation, see Ref. [3]: λ_0 is called a Hopf bifurcation point at the zero equilibrium if for any sufficiently small $\varepsilon>0$ there is a parameter value $\lambda_\varepsilon\in(\lambda_0-\varepsilon,\lambda_0+\varepsilon)$ such that Eq. (1) has a nonstationary periodic solution with an amplitude $\|z\|_C<\varepsilon$.

Theorem 1. $\lambda_0=0$ *is a Hopf bifurcation point at the zero equilibrium for Eq.* (2).

If in addition $\mu(0,0)>0$, then it can be shown that for any sufficiently small $\lambda>0$ Eq. (2) has a cycle with an amplitude $r(\lambda)=\kappa\lambda+o(\lambda)$, $\lambda\to 0$. Note that this asymptotics differs from the asymptotics $r(\lambda)=\kappa\sqrt{\lambda}+o(\sqrt{\lambda})$ of the cycle amplitude of the classical van der Pol equation (see, e.g. Ref. [4]).

Another type of the Hopf bifurcation is observed at infinity. Similarly to the previous definition, λ_0 is called a Hopf bifurcation point at infinity for Eq. (1) with $f=f(z,\lambda)$ if for every $\varepsilon>0$ there is a parameter value $\lambda_\varepsilon\in(\lambda_0-\varepsilon,\lambda_0+\varepsilon)$ such that Eq. (1) has a cycle with an amplitude $\|z\|_C>\varepsilon^{-1}$ (see Ref. [5]).

Theorem 2. $\lambda_0=0$ *is a Hopf bifurcation point at infinity for Eq.* (2).

The asymptotics of the amplitude $r(\lambda)$ of the large cycles for small $\lambda>0$ depends on the asymptotic behaviour of the measure density μ of the Preisach operator. For example, if the density has a bounded support, then $r(\lambda)=\gamma\lambda^{-1/2}+o(\lambda^{-1/2})$, see Fig. 1(a) and (b).

Under certain conditions, we can guarantee that the local branches of cycles, originating from zero and from infinity at $\lambda=\lambda_0$, are connected. We say that Eq. (1) with $f=f(z,\lambda)$ has a continuous branch of cycles, connecting zero and infinity at $\lambda=\lambda_0$, if for any $r>0$ this equation has a cycle z_r of the amplitude $r=\|z_r\|_C$ for some parameter value $\lambda=\lambda(r)$, the cycles z_r and their periods depend continuously on r, the function $\lambda(r)$ is continuous, and $\lambda(r)\to\lambda_0$ as $r\to 0$, $r\to\infty$.