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A rigorous treatment of nucleation modes spectrum in micromagnetics

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Abstract

The nucleation problem for an anisotropic ellipsoidal magnetic particle is studied in the framework of micromagnetics. The stability of a spatially uniform micromagnetic equilibrium is connected to the positive definitiveness of a quadratic functional which is associated to an appropriate self-adjoint integro-differential operator. The spectrum of this operator is studied rigorously by using the theory of continuous and weakly coercive bilinear forms.

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1. Introduction

In general terms, the problem of nucleation in magnetic bodies consists in the study of the stability of a magnetization equilibrium when the amplitude of the external field is changed. The main goal of the analysis is to compute the critical values of the applied field such that the magnetization equilibrium becomes unstable. In this paper, we will limit ourselves to the classical case of uniformly magnetized particles. This case has been studied in detail by using micromagnetics and it has been shown that depending on the geometry of the particle, and on the type and strength of anisotropy, one can observe various magnetization instability spatial patterns (uniform rotation, curling, buckling, etc.) [1–3] which are usually referred to as ''nucleation modes''. The problem of determining these modes can be formulated as an eigenvalue problem and most studies have been focused on the determination of special eigenvalues and eigenfunctions. In this paper, on the other hand, we prove rigorous results on the general properties of the nucleation modes spectrum. The main results are that the nucleation spectrum is discrete in nature and nucleations modes form a complete set of orthogonal functions.

2. Formal theory of micromagnetic equilibria and stability

We start our discussion by recalling basic facts about micromagnetic theory. This theory is based on the micromagnetic Gibbs free energy functional associated with the magnetic body. In dimensionless form, this functional can be written as

$$\mathcal{G}[\boldsymbol{m}] = \int_\Omega \left[\frac{1}{2}(\nabla \boldsymbol{m})^2 + \varphi(\boldsymbol{m}) - \boldsymbol{h}_\mathrm{a}\cdot\boldsymbol{m} - \frac{1}{2}\boldsymbol{h}_\mathrm{M}[\boldsymbol{m}]\cdot\boldsymbol{m}\right] \mathrm{d}V, \tag{1}$$

where $\boldsymbol{m} = \boldsymbol{m}(\mathbf{r})$ is the magnetization vector field, Ω is the region occupied by the body, $\varphi : \mathbb{R}^3 \to \mathbb{R}$ is a function representing the anisotropy energy, $(\nabla \boldsymbol{m})^2 = |\mathbf{grad}\, m_x|^2 + |\mathbf{grad}\, m_y|^2 + |\mathbf{grad}\, m_z|^2$ (where $|\cdot|$ denotes the usual Euclidean norm), and $\boldsymbol{h}_\mathrm{a}$ is a given external applied field. The vector field $\boldsymbol{h}_\mathrm{M}[\boldsymbol{m}]$ is the magnetostatic field generated by the magnetized body and it can be expressed as $\boldsymbol{h}_\mathrm{M}[\boldsymbol{m}] = \mathbf{grad}\,\mathrm{div}\,\psi[\boldsymbol{m}]$, where

$$\psi[\boldsymbol{m}] = \frac{1}{4\pi}\int_\Omega \frac{\boldsymbol{m}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V(\boldsymbol{r}'). \tag{2}$$

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The fundamental assumption of micromagnetic theory is that the only admissible magnetization vector fields are those that fulfill the following constraint:

$$|\boldsymbol{m}(\mathbf{r})| = 1 \quad \forall \mathbf{r} \in \Omega. \tag{3}$$

Micromagnetic equilibria correspond to vector fields $\boldsymbol{m}$ which fulfill the constraint (3) and are such that the first derivative of $\mathscr{G}[\boldsymbol{m}]$ vanishes. Stability of equilibria can be then studied by considering higher order derivatives. To compute these derivatives, we have first to define the class of admissible variations of vector field compatible with the constraint (3). In this respect, we notice that variations $\delta\boldsymbol{m}(\mathbf{r})$ of $\boldsymbol{m}(\mathbf{r})$ such that $|\boldsymbol{m}(\mathbf{r}) + \delta\boldsymbol{m}(\mathbf{r})| = 1$ can be obtained by performing a rotation of $\boldsymbol{m}(\mathbf{r})$ at each spatial location. The most general rotation can be generated by using the 3×3 matrix $\Lambda(\boldsymbol{\vartheta})$ such that $\Lambda(\boldsymbol{\vartheta}) \cdot \mathbf{u} = \boldsymbol{\vartheta} \times \mathbf{u}$, and considering the matrix exponential $\exp(\Lambda(\boldsymbol{\vartheta}))$. Indeed, the matrix product $\exp(\Lambda(\boldsymbol{\vartheta})) \cdot \boldsymbol{m}$ produces a rotation of $\boldsymbol{m}$ of an angle $\vartheta = |\boldsymbol{\vartheta}|$ around the axis identified by the unit vector $\boldsymbol{\vartheta}/|\boldsymbol{\vartheta}|$ ($|\cdot|$ denotes the $\mathbb{R}^3$ Euclidean norm). Thus, the admissible variations are given by $\boldsymbol{m}(\mathbf{r}) + \delta\boldsymbol{m}(\mathbf{r}) = \exp(\Lambda(\boldsymbol{\vartheta}(\mathbf{r}))) \cdot \boldsymbol{m}(\mathbf{r})$. With this definition of admissible variations we can compute the first derivative of $\mathscr{G}[\boldsymbol{m}]$ around a given vector field $\boldsymbol{m}_0$. By using appropriate algebraic manipulations of Eqs. (1)–(3), one obtains the following expression:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\{\mathscr{G}[\exp(\Lambda(\varepsilon\boldsymbol{\vartheta})) \cdot \boldsymbol{m}_0] - \mathscr{G}[\boldsymbol{m}_0]\} = -(\boldsymbol{m}_0 \times \boldsymbol{h}_{\text{eff}}[\boldsymbol{m}_0], \boldsymbol{\vartheta})_\Omega + \int_{\partial\Omega} \boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial \boldsymbol{n}} \cdot \boldsymbol{\vartheta}\, \mathrm{d}S, \tag{4}$$

where $\boldsymbol{h}_{\text{eff}}[\boldsymbol{m}]$ is the effective field and it is given by

$$\boldsymbol{h}_{\text{eff}}[\boldsymbol{m}] = \Delta\boldsymbol{m} + \frac{\partial\varphi}{\partial\boldsymbol{m}} + \boldsymbol{h}_{\text{M}}[\boldsymbol{m}] + \boldsymbol{h}_{\text{a}}, \tag{5}$$

Δ denotes the Laplacian operator, and $(\boldsymbol{u}, \boldsymbol{w})_\Omega = \int_\Omega \boldsymbol{u}(\mathbf{r}) \cdot \boldsymbol{w}(\mathbf{r})\, \mathrm{d}V$ is the usual $\mathbb{L}^2$ scalar product, $\partial\varphi/\partial\boldsymbol{m}$ is the gradient of φ with respect to $\boldsymbol{m}$, and $\partial\Omega$ is the boundary of Ω. From Eq. (4), and using the arbitrariness of $\boldsymbol{\vartheta}(\mathbf{r})$, we can conclude that micromagnetic equilibria are given by the conditions (Brown's equations [1]):

$$\boldsymbol{m}_0 \times \boldsymbol{h}_{\text{eff}}[\boldsymbol{m}_0] = 0 \text{ in } \Omega, \quad \frac{\partial \boldsymbol{m}_0}{\partial \boldsymbol{n}} = 0 \text{ on } \partial\Omega, \tag{6}$$

where $\boldsymbol{n}$ is the unit normal to $\partial\Omega$. Eqs. (6) have to be complemented by the constraint in Eq. (3).

We study now the nucleation problem [1] for uniformly magnetized particles. In this respect, we make the following assumptions: (a) Ω is an ellipsoid with principal axes along the Cartesian unit vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$, respectively; (b) the anisotropy is uniaxial with $\varphi(\boldsymbol{m}) = -(\kappa/2)m_z^2$ (κ is the anisotropy constant and $\mathbf{e}_z$ is the easy axis); (c) the applied field is spatially uniform and $\boldsymbol{h}_\text{a} = h_{\text{a}z}\mathbf{e}_z$. In this case, one has that $\boldsymbol{m}_0 = \mathbf{e}_z$ is always a spatially uniform equilibrium. This is consequence of the fact that $\boldsymbol{h}_\text{M}[\mathbf{e}_z] = -N_z\mathbf{e}_z$, where N_z is z-demagnetizing factor of the ellipsoid, and thus $\boldsymbol{h}_{\text{eff}}[\mathbf{e}_z] = h_0\mathbf{e}_z$, where $h_0 = (\kappa - N_z + h_{\text{a}z})$.

We want now to study the stability of the equilibrium $\mathbf{e}_z$. To this end, by using suitable algebraic manipulations, one can derive the following expansion:

$$\mathscr{G}[\exp(\Lambda(\varepsilon\boldsymbol{\vartheta})) \cdot \mathbf{e}_z] = \mathscr{G}[\mathbf{e}_z] + \frac{\varepsilon^2}{2}\{a[\boldsymbol{v}, \boldsymbol{v}] + h_0(\boldsymbol{v}, \boldsymbol{v})_\Omega\} + \mathscr{O}(\varepsilon^3), \tag{7}$$

where $\boldsymbol{v} = \boldsymbol{\vartheta} \times \mathbf{e}_z$ (thus $\boldsymbol{v} \cdot \mathbf{e}_z = 0$), and $a[\boldsymbol{u}, \boldsymbol{w}]$ is the positive definite symmetric bilinear form given by

$$a[\boldsymbol{u}, \boldsymbol{w}] := \sum_h (\textbf{grad}\, u_h, \textbf{grad}\, w_h)_\Omega - (\textbf{grad}\,\text{div}\,\psi[\boldsymbol{u}], \boldsymbol{w})_\Omega. \tag{8}$$

It is important to underline that in Eq. (7) the vector field $\boldsymbol{v}$ has zero component along $\mathbf{e}_z$. This is an important constraint of our problem which has to be taken into account. In this respect, in the following, we will tacitly assume that all vector fields denoted by $\boldsymbol{v}$, $\boldsymbol{u}$, and $\boldsymbol{w}$ are in the subspace of vector field which have zero z-components. This observation applies also to Eq. (8). The symmetry of the second term in Eq. (8) can be demonstrated by using the magnetostatic reciprocity theorem [1] which, in our notations, reads

$$(\textbf{grad}\,\text{div}\,\psi[\boldsymbol{u}], \boldsymbol{w})_\Omega = (\boldsymbol{h}_\text{M}[\boldsymbol{u}], \boldsymbol{w})_\Omega = (\boldsymbol{u}, \boldsymbol{h}_\text{M}[\boldsymbol{w}])_\Omega. \tag{9}$$

The positive nature of $a[\boldsymbol{u}, \boldsymbol{u}]$ is obtained by the positivity of magnetostatic energy:

$$-(\textbf{grad}\,\text{div}\,\psi[\boldsymbol{u}], \boldsymbol{u})_\Omega = -(\boldsymbol{h}_\text{M}[\boldsymbol{u}], \boldsymbol{u})_\Omega = \int_{\mathbb{R}^3} |\boldsymbol{h}_\text{M}[\boldsymbol{u}]|^2\, \mathrm{d}V, \tag{10}$$

and from the fact that constant vector field (i.e. with zero gradient) have always a strictly positive magnetostatic energy.

By using the thermodynamic principle that stable equilibria are minima of the free energy of the system, one can say that the stability of $\mathbf{e}_z$ is related to the positive definitiveness of the quadratic form $\{a[\boldsymbol{v}, \boldsymbol{v}] + h_0(\boldsymbol{v}, \boldsymbol{v})_\Omega\}$ (see Eq. (7)) and this is related to the value of the scalar h_0. Indeed, for sufficiently negative values of $h_0 = (\kappa - N_z + h_{\text{a}z})$ and thus for appropriately low or negative values of $h_{\text{a}z}$ the equilibrium $\mathbf{e}_z$ becomes unstable. The precise value of $h_{\text{a}z}$ which produce instability can be determined by studying the spectrum of the self-adjoint operator associated to the symmetric form $a[\boldsymbol{u}, \boldsymbol{w}]$. Indeed, by using integration by parts one can prove that $a[\boldsymbol{u}, \boldsymbol{w}] = (\mathscr{C}[\boldsymbol{u}], \boldsymbol{w})_\Omega + \int_{\partial\Omega} \boldsymbol{w} \cdot \partial\boldsymbol{u}/\partial\boldsymbol{n}\, \mathrm{d}S$, where

$$C[\boldsymbol{u}] := -\Delta\boldsymbol{u} - \mathscr{P}_\perp \boldsymbol{h}_\text{M}[\boldsymbol{u}], \tag{11}$$

and $\mathscr{P}_\perp(\boldsymbol{h}) = h_x\mathbf{e}_x + h_y\mathbf{e}_y$. The operator in Eq. (11), complemented by the boundary condition $\partial\boldsymbol{u}/\partial\boldsymbol{n} = 0$ on $\partial\Omega$, is a self-adjoint operator with respect to the usual $\mathbb{L}^2$ scalar product (in the subspace of vector fields with zero z-component). As a consequence of the discussion above, one can conclude that the operator above has real and positive eigenvalues. Further conclusions on the spectrum of this operator requires more careful reasonings which are reported below.