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Rigged strings, Bethe Ansatz, and the geometry of the classical configuration space of the Heisenberg magnetic ring

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Abstract

Composition of two combinatoric algorithms, Robinson–Schensted (RS) and Kerov–Kirillov–Reshetikhin (KKR), defines the bijection which maps the set of all magnetic configurations of the Heisenberg ring of N nodes with the spin $\frac{1}{2}$ onto that of all exact Bethe Ansatz (BA) eigenstates of the nearest neighbour isotropic Hamiltonian. We point out here that this bijection allows one to predict all quantum numbers of an exact BA eigenstate (encoded as rigged string configurations of KKR) from each magnetic configuration. Moreover, this bijection provides completeness of solutions on each orbit of the symmetric group of N nodes, acting on the set of all magnetic configurations. Each such an orbit can be interpreted as the classical configuration space of the system of r Bethe pseudoparticles—spin deviations which are hard-core objects, moving on the magnetic ring by jumps to non-occupied nearest neighbours. Such an interpretation provides a transparent and combinatorially unique description of all exact BA eigenstates in terms of magnetic configurations—the initial basis for quantum calculations. Within this picture, an l-string introduced by Bethe corresponds to an extended object, consisting of $2l$ consecutive nodes: first l spin deviations, and then l nodes with the spin projection $+\frac{1}{2}$, all bounded together and put somewhere inside the magnetic chain. Each BA solution is a rigged string configuration, i.e. a distribution of a number q, $0 \leqslant q \leqslant r$, of such objects inside the chain. Allowed distributions are subjected to certain combinatoric restrictions (rules of navigation), expressed in terms of l-holes and riggings, and presented graphically as paths, associated with schemes of consecutive coupling of N spins $\frac{1}{2}$ along the RS algorithm. We discuss here thoroughly some implications of existence of such a bijection. In particular, we demonstrate the way in which structure of orbits of the translation group C_N on the classical configuration space for r Bethe pseudoparticles imposes the corresponding arrangement of rigged string configurations. Essentially, within a single cycle along a C_N-orbit, each l-string moves from the left to the right with increase of its rigging by one unit until reaching the last node N of the ring, then shortens its length to zero, and next arises at the first (leftmost) node and elongates up to its previous length l. Such considerations allow us also to discuss the geography of rigged string configurations on the classical configuration space. In particular, we put emphasis on the fact that—within this combinatoric picture—the l-strings originate from corresponding sizes of islands of consecutive Bethe pseudoparticles in the classical configuration space. Thus, the set of all magnetic configurations with r spin deviations acquires the interpretation of an r-dimensional manifold with F-dimensional boundaries, the integer F, $1 \leqslant F \leqslant r$, being the number of islands of consecutive Bethe pseudoparticles. Each magnetic configuration belonging to the generic part $F = r$ yields only a 1-string under considered here bijection, whereas l-strings with $l > 1$ arise from islands located in appropriate boundaries.

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1. Introduction

It is well known that the exact solutions of the one-dimensional Heisenberg magnetic ring, given by the famous Bethe Ansatz (BA) [1], are classified in terms of

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Takahashi [2] integers or half-integers, which combinatorially correspond to *rigged string configurations* [3–7]. Rigged string configurations are physically interpreted as a collection of strings, i.e. bound states of a number of spin deviations (Bethe pseudoparticles), each string equipped with a definite quasimomentum, expressed by the Takahashi number which varies over a definite range. In this way, each eigensolution of the Heisenberg Hamiltonian for the ring of N spins $s=\frac{1}{2}$ is fully characterized by the pair $(\nu, \mathscr{L})$, where, according to the terminology proposed by Kerov, Kirillov and Reshetikhin (KKR) [4], $\nu = (m_1, m_2, \ldots)$ is the *string configuration*, with m_l being the number of strings of the length $l = 1, 2, \ldots,$ and $\mathscr{L}$ is the corresponding collection of integers which label quasimomenta of strings, referred to as the *rigging* of ν. The set of all BA eigensolutions is, therefore, decomposed into subsets with the same string configuration, which differ by the distribution of quasimomenta over strings.

We recall briefly that the space $\mathscr{H}$ of all quantum states of the Heisenberg magnet, with $\dim \mathscr{H} = 2^N$, decomposes as

$$\mathscr{H} = \sum_{r=0}^{N} \oplus \mathscr{H}^r \tag{1}$$

into subspaces $\mathscr{H}^r$, $\dim \mathscr{H}^r = \binom{N}{r}$, with the fixed number r of Bethe pseudoparticles (spin deviations). Stated otherwise, $\mathscr{H}^r$ is the space of all states of the magnet with the projection

$$M = \frac{N}{2} - r \tag{2}$$

of the total spin, or of all states with the *weight*

$$\mu = \{N - r, r\}. \tag{3}$$

Clearly, $\mathscr{H}^r$ and $\mathscr{H}^{N-r}$ are particle–hole counterparts, so we restrict in the sequel to $r \leqslant N/2$. Then, each space $\mathscr{H}^r$ decomposes as

$$\mathscr{H}^r = \sum_{S=N/2-r}^{N/2} \oplus \mathscr{H}^{rS} \tag{4}$$

into subspaces $\mathscr{H}^{rS}$ with the total spin S (and $M = N/2 - r$). By the duality of Weyl [8] between the actions of the symmetric group $\sum_N$ and the unitary group $U(n)$, $n = 2s+1 = 2$, in the space $\mathscr{H}$ (cf., e.g. Refs. [9,10] for detail), it defines the partition λ of the integer N, of the *shape*

$$\lambda = \{N - r', r'\}, \tag{5}$$

such that λ labels the irrep Δ^λ of $\sum_N$,

$$\dim \Delta^\lambda = \dim \mathscr{H}^{rS}, \tag{6}$$

and

$$S = \frac{N}{2} - r', \quad 0 \leqslant r' \leqslant r. \tag{7}$$

The extreme case

$$S_{\text{h.w.}} = \frac{N}{2} - r = M \tag{8}$$

is usually referred to as *the highest weight* subspace in $\mathscr{H}^r$. KKR [4] defined a bijection between the set of all rigged string configurations which span a space $\mathscr{H}^{rS}$ as exact solutions of BA, and the set SYT(λ) of all standard Young tableaux of the shape λ, which provides a basis in $\mathscr{H}^{rS}$ along the duality of Weyl, that is, the orthogonal Young basis in $\mathscr{H}^{rS}$, treated as the carrier space of the irrep Δ^λ of Σ_N (cf., e.g. Ref. [9]). Moreover, they pointed out for another bijection, provided by the Robinson [11]–Schensted [12] (RS) algorithm (cf. also Refs. [9,10,13,14]), which relates the irreducible basis of the duality of Weyl with the basis of all magnetic configurations—the initial stage for all quantum calculations in the space $\mathscr{H}$.

The setting provided by KKR involves the general scheme of the duality of Weyl for an arbitrary $n = 2s+1$, and is therefore applicable—after an appropriate generalization of the Heisenberg Hamiltonian—to an arbitrary spin s. In this paper, we confine ourselves to the case $s = \frac{1}{2}$, with an appropriate restriction of definitions and notation (in particular, we omit the notion of a *colour* of a string, which is redundant here).

In the present paper, we aim to point out a somehow surprising conclusion, stemming from existence of the two bijections, RS and KKR. Solutions of highly non-linear BA equations involve a complete set of exact quantum numbers for each eigenstate. The combination of the two above bijections implies that these tiny results are already coded, in a combinatorially bijective way, in the set of all magnetic configurations. Thus, each *single* magnetic configuration anticipates a definite eigenstate, which is, as a rule, a highly correlated wavepacket of a number of magnetic configurations. In other words, the results of algebraic calculations of BA equations are predicted and classified by purely combinatoric methods. We point out in this paper some evident conclusions which emerge from this fact.

We also aim to examine properties of the composition, RSKKR = KKR ∘ RS, of these two bijections, under the action of the cyclic group C_N—the translational symmetry group of the problem, along the description of the basis of wavelets proposed in our earlier paper [15]. We have observed [15,16] that the set $Q^{(r)}$ of all those magnetic configurations which span unitarily the space $\mathscr{H}^r$ has a natural interpretation of the classical configuration space of the system of r Bethe pseudoparticles—indistinguishable hard-core particles moving on the magnetic ring. In particular, we have shown that the geometry imposed on $Q^{(r)}$ by the dynamics resulting from the Heisenberg Hamiltonian, and by the action of the translation group C_N, yields a curvature of the manifold $\mathscr{M}^{(r)}$ into which the discrete set $Q^{(r)}$ can be regularly embedded. Such a curvature results simply from the observation that each C_N-orbit in $Q^{(r)}$ should be a loop. Here, we investigate