Physica C 469 (2009) 732–735

Contents lists available at ScienceDirect

Physica C

journal homepage: www.elsevier.com/locate/physc

Perpendicular susceptibility of completely shielded elliptical and rectangular superconducting films

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ARTICLE INFO

Article history:
Received 5 March 2009
Received in revised form 25 March 2009
Accepted 2 April 2009
Available online 9 April 2009

PACS:
41.20.Gz
74.25.Ha
74.78.−w

Keywords:
Magnetic shielding
Susceptibility
Superconducting film
Demagnetizing factor

ABSTRACT

The perpendicular susceptibility $-\chi_0$ of completely shielded elliptical and rectangular superconducting films with different aspect ratios has been calculated accurately. The obtained χ_0 may be compared with the measured low-field limit of ac susceptibility to check the quality of superconducting films, and used as a scaling parameter to obtain field amplitude dependent complex critical-state ac susceptibility.

1. Introduction

Perfect diamagnetism is a fundamental property of any superconductor of large dimensions compared with the London penetration depth. This means that when an enough small field H_a is applied to the superconductor, its internal flux density $B = 0$ or equivalently, the material susceptibility $\chi = -1$ [1]. The field may be applied at a high temperature and then the sample is cooled to the superconducting state (the Meissner effect) or applied after the sample is zero-field cooled to the superconducting state (the shielding effect). The same susceptibility occurs for both cases at the same temperature, so that such a diamagnetism may be studied quantitatively by low-frequency ac susceptibility measurements if the susceptibility of a completely shielded body, $-\chi_0$, with the same shape and dimensions as those of the measured sample is accurately calculated. The present work is devoted to such calculations for thin films when H_a is applied perpendicular to the film surface.

Exact χ_0 has been calculated analytically for a thin disk of radius b and thickness t and an infinitely long thin strip of width $2b$ and thickness t [2–4]; the results are

$$\chi_0 = \frac{8b}{3\pi t} \text{ (disk)}, \tag{1}$$

$$\chi_0 = \frac{\pi b}{2t} \text{ (strip)}. \tag{2}$$

We will show that both results may be derived from the known demagnetizing factor of relevant ellipsoids, as already been described in [5]. Using this technique, χ_0 of elliptical films of different values of semiaxes a and b and thickness t will be calculated exactly in Section 2.

Less accurate values (with errors on the order of 1%) of numerically calculated χ_0 for rectangular films were given in [6]. Recently, a new procedure has been proposed, with which χ_0 of a square film is calculated to a precision of 0.1% [7]. Extending the work in [7], we will calculate χ_0 up to a similar precision for rectangular films of dimensions of $2a \times 2b \times t$ in Section 3.

Although elliptical films of $a \neq b$ may not be of practical importance, their results can be compared with those of rectangular films, and, as described below, treating the problem of elliptical films is a good preparation to optimize the calculation conditions of the numerical calculations for rectangular films.

It has to be commented that owing to strong demagnetizing effects, type-I or type-II superconducting films will present an intermediate or vortex state, respectively, but not a Meissner state in a

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doi:10.1016/j.physc.2009.04.002

perpendicular H_a even if H_a is very low [1]. Thus, the calculated $-\chi_0$ for the completely shielded state should be compared with the measured low-frequency ac susceptibility after extrapolation to $H_a = 0$. Moreover, the effect of the two-dimensional screening length has to be taken into account in such a comparison [3,4,8]. If the film is a hard type-II superconductor, an accurate χ_0 may not only be used as a standard for quality check of the film, it may also be used to check the validity of a scaling law proposed by Gilchrist [9], by which the field dependent critical-state ac susceptibility of films of different shape may be obtained from the analytical results of thin disk or strip, as described in [7].

2. Susceptibility of elliptical films

There is a relation between χ_0 and the demagnetizing factor N of a completely shielded ellipsoid when H_a is applied along a principal axis [10]:

$$\chi_0 = \frac{1}{1-N}. \tag{3}$$

This equation may be used for obtaining the perpendicular χ_0 of an elliptical film of semiaxes a and b and thickness t from the known N value of an ellipsoid of semiaxes a, b, and c, with

$$c = 3t/4, \tag{4}$$

so that its volume is the same as that of the film. If $a \geqslant b \geqslant c$, the exact formula for N of this ellipsoid is [11,12]

$$N = \frac{\cos\phi\cos\theta}{\sin^3\theta(1-k^2)}\left[\frac{\sin\theta\cos\phi}{\cos\theta} - E(k,\theta)\right], \tag{5}$$

where $E(k,\theta)$ is the second kind elliptical integral and

$$\cos\theta = \frac{c}{a}, \quad \cos\phi = \frac{b}{a}, \quad k = \frac{\sin\phi}{\sin\theta}. \tag{6}$$

Substituting N in Eq. (3) by Eq. (5), where c is replaced by Eq. (4), and considering $c/a \to 0$, one obtains for the film that

$$\chi_0 = \frac{4b}{3tE(k)}, \tag{7}$$

where $E(k)$ is the second kind complete elliptical integral.

The above derivation may be explained as follows. Assume that the ellipsoid is centered at the origin with its semiaxes a, b, and c lying along the x, y, and z axes, respectively. When H_a is applied in the positive z direction, magnetization M and field H are directed to the negative and positive z direction, respectively, and both are uniform inside the body. The condition of complete shielding $B_z = \mu_0(M+H) = 0$ is realized by a certain distribution of magnetic moment density $g(x,y) = Mt(x,y)$, where $t(x,y)$ is the local thickness of the ellipsoid along the z axis. When $c/a \to 0$, the same internal $B_z = 0$ may be realized for the thin film at the same H_a with the same moment density $g(x,y)$. Thus, the same total moment m occurs for both cases, and χ_0 defined by

$$\chi_0 \equiv \frac{-m}{H_aV}, \tag{8}$$

where V is the volume of the body, is also the same if both bodies have the same V, so that Eq. (4) holds and Eq. (7) is obtained for the film.

Eq. (1) for a circular disk may be derived from Eq. (7) when $a/b = 1$. It is also derived from Eq. (7) for an infinitely long elliptical film that

$$\chi_0 = \frac{4b}{3t} \quad (a/b \to \infty). \tag{9}$$

Numerical results of $\chi_0 t/(2b)$ of elliptical films for several values of a/b are listed in Table 1 and plotted in Fig. 1.

Table 1
Perpendicular $\chi_0 t/(2b)$ of elliptical films of semiaxes a and b and thickness t and rectangular films of dimensions $2a \times 2b \times t$.

a/b	$\chi_0 t/(2b)$ (ell.)	$\chi_0 t/(2b)$ (rec.)
0.7143	0.3512	0.3771
1	0.4244	0.4547
1.4	0.4917	0.5279
2	0.5505	0.5940
3	0.5986	0.6533
5	0.6346	0.7042
7	0.6478	0.7271
10	0.6562	0.7442
20	0.6634	0.7643
∞	0.6667	0.7854

3. Susceptibility of rectangular films

Unlike the case of elliptical films whose exact χ_0 may be calculated analytically, the χ_0 for rectangular films has to be calculated numerically, except when $a/b \to \infty$, for which the exact Eq. (2) has been derived. In the present work, the film is divided into equal rectangular elements and a magnetic energy minimization procedure is used for numerical computations [13,14,7].

It may be realized from the above approach for calculating χ_0 of elliptical films that the correct $g(x,y)$ for the completely shielded elliptical film is proportional to the local thickness $t(x,y)$ of its equivalent ellipsoid; $g(x,y)$ has negative values and presents a broad minimum in the central region and $g(x,y) = 0$ occurs at the edge with an infinite gradient, which corresponds to an infinite screening current density J. The numerical calculations for rectangular films may be optimized based on these observations.

In general, the rectangular film of sides $2a$ and $2b$ may be divided into $n_a \times n_b$ rectangular cells, each having an area of $A_c = 4ab/(n_a n_b)$. Knowing that $g(x,y)$ for the elliptical case is proportional to $t(x,y)$, which has elliptical isometric loops of aspect ratio of a/b, an optimum condition of $n_a = n_b$ with $A_c = 4ab/n_a^2$ should be applied for the numerical calculations. Under this condition, the correct $g(x,y)$ function for the film is approximated by an $(n_a+1)\times(n_a+1)$ array of $g(i,j)$, where indices $i,j = 0,1,2,\ldots,n_a$ correspond to the node positions and $g(i,j) = 0$ holds if $i,j = 0$ or n_a. Applying a field H_a in the positive z direction, the moment $g(i,j)A_c$ is always negative to ensure internal $B_z(i,j) = 0$ everywhere. This condition is approximately satisfied by executing the magnetic energy minimization procedure mentioned above. Starting with $g(i,j)A_c = 0$, a positive value of H_a is applied and a position (i,j), at which decreasing $g(i,j)A_c$ by a given negative moment increment $\Delta m \equiv \Delta g A_c$ would yield the maximum decrease in the magnetic energy, is looked for. When this place is found the $g(i,j)A_c$ is changed by adding Δm. This process is repeated until the energy cannot be further minimized, so that a set of $g(i,j)A_c$ is found for this value of H_a, from which χ_0 is calculated by

$$\chi_0 = \frac{-A_c}{H_a 4abt}\sum_{i,j=0}^{n_a} g(i,j) = \frac{-1}{H_a n_a^2 t}\sum_{i,j=0}^{n_a} g(i,j). \tag{10}$$

In the program four Δm's were added to four symmetric nodes each time. Detailed computations are explained elsewhere [15]. It has been found that the calculated $\chi_0 t/(2b)$ decreases with increasing H_a and decreasing n_a. For a fixed n_a, it has a linear relation with $1/H_a$ when H_a is high enough, as shown for $a/b = 1$ in [7]. Another example is given in Fig. 2a for the case of $a/b = 5$. Linearly extrapolating to $1/H_a = 0$ to reduce the discretization error owing to finite Δm, one gets a $\chi_0 t/(2b)$ for each value of n_a. Drawing the extrapolated $\chi_0 t/(2b)$ against $1/n_a$, another linear relation is found, as