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Analytic expressions for critical-state ac susceptibility of rectangular superconducting films in perpendicular magnetic field

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ABSTRACT

A set of analytic expressions for the critical-state magnetization and ac susceptibility $\chi' - j\chi''$ curves of rectangular superconducting films of arbitrary aspect ratio in perpendicular field is obtained. They are formulated based on the exact/accurate high/low-field limits for rectangular films together with the exact analytic formulas for thin circular disks. Thus, the accuracy of the expressions is very high in high and low fields. At a characteristic intermediate field amplitude, where χ'' takes its maximum, the error is checked by direct numerical calculations to be well below 1%. Thus, the provided expressions can be conveniently and satisfactorily used for practical and theoretical researches of superconducting materials, such as coated conductors.

1. Introduction

The properties of type-II superconducting films of thickness on the order of London penetration depth may be studied by ac susceptibility technique. If the field is applied parallel to the film surface, besides the fact that the detection of small magnetic moment requires an extremely high instrument sensitivity, the measured susceptibility is dominated by the penetrated Meissner currents but not the practically more important critical currents. In contrast, in the perpendicular case, a small ac field will result in a large magnetic moment owing to a large demagnetizing factor accompanied by a negative low-field susceptibility, the effect of London penetration depth is very weak, and the electromagnetic state of the measured sample is a counterpart of that for the transport measurements with self-field. As a consequence, although the critical current and ac losses of thin superconducting tapes (for example, coated conductors) can be easily tested by transport measurements, their performance may be studied more completely by additional measurements of the ac susceptibility of a rectangular sample in perpendicular magnetic field.

In order to get important information from complex ac susceptibility, $\chi = \chi' - j\chi''$, measurements, experimental data have to be compared with theoretical χ as a function of the ac field amplitude H_m calculated from the critical-state (CS) model. Such calculations were originally carried out analytically for longitudinal geometries, for which the demagnetizing effect may be neglected. The CS calculations for long rectangular bars of different values of aspect ratio may be found in [1–3], where the field dependence of J_c is also considered. For comparing with experimental results, numerical CS calculations even with a complex field and position dependent J_c are also not difficult [4]. In the perpendicular case, however, the demagnetizing effect is extremely strong, and analytical CS calculations may be done only for limited geometries with a constant J_c, whereas numerical CS calculations for other geometries are extremely time consuming even for a constant J_c. Thus, it will be of great importance if an approximate analytic solution can be found based on existing results.

Actually, a scaling law was proposed by Gilchrist [5], which relates the field amplitude H_m at which χ'' takes its maximum, $H_m(\chi''_m)$, to the low-field limit of $\chi', -\chi_0$, and saturation magnetization M_s of films of any shape with a constant J_c. This scaling law has been refined by Chen et al. [6].

The scaling law is built up based on the exact analytic derivations for a thin circular disk and an infinitely long thin strip [7–11]. An approximate but very accurate analytic solution for the initial magnetization curve of elliptical superconducting films has been derived by Mikitik and Brandt [12]. By numerical evaluation, only a maximum difference of 0.013 in normalized magnetization M/M_s is found among the magnetization curves for different values of aspect ratio, which justifies the accuracy of the scaling law for elliptical films. Since rectangular films have four right-angle corners that elliptical films do not have, a further validation of the scaling law for rectangular films is necessary. Such a validation was partially carried out for a square film with positive result [6]. In the present work, we will complete and extend the validation for rectangular films, so that a set of accurate analytic

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expressions for the CS magnetization and ac susceptibility curves of rectangular films is formulated.

The contents are arranged as follows. As the basis of formulating the expressions for rectangular films, the analytic solution for thin disks and the scaling law are reviewed in Section 2. The formulas for two scaling parameters, saturation magnetization and initial susceptibility, for rectangular films are given in Section 3. The numerical calculations for CS magnetization and ac susceptibility of rectangular films are described in Section 4. The analytical expressions for CS magnetization and ac susceptibility of rectangular films are summarized in Section 5. Some conclusions are presented in Section 6.

2. Solution for thin disks and scaling law for films

2.1. Analytic solution for thin disks

As the basis of the analytic expressions of CS magnetization and ac susceptibility of rectangular superconducting films in perpendicular magnetic field, we review the analytic solution for thin disks derived in [9]. For a superconducting thin disk of diameter d, thickness t, and critical-current density J_c placed in a perpendicularly applied field H, magnetization M is expressed by

$$M = -\chi_0 H S(x), \tag{1}$$

where $-\chi_0$ is the initial susceptibility $(dM/dH)_{H=0}$ and

$$S(x) = \frac{1}{2x}\left[\arccos\left(\frac{1}{\cosh x}\right) + \frac{\sinh x}{\cosh^2 x}\right] \tag{2}$$

with

$$x = 2H/(J_c t). \tag{3}$$

Analytic expressions of hysteresis $M(H)$ loop for $-H_m \leqslant H \leqslant H_m$ can be obtained from these equations. Assuming a sinusoidally varying H between $\pm H_m, H_m > 0$ being ac field amplitude, $\chi(H_m)$ is obtained by Fourier analysis of the hysteresis loop as

$$\chi' = \frac{2\chi_0}{\pi}\int_0^{\pi}(1-\cos\theta)S(y)\cos\theta d\theta, \tag{4}$$

$$\chi'' = \frac{2\chi_0}{\pi}\int_0^{\pi}[(1-\cos\theta)S(y) - S(x)]\sin\theta d\theta, \tag{5}$$

where

$$x = 2H_m/(J_c t) \tag{6}$$

and $S(y)$ is expressed by Eq. (2) where x is replaced by $y = (1-\cos\theta)x/2$ with the new x expressed by Eq. (6).

The normalized magnetization $-M/(J_c d)$ versus the normalized field $H/(J_c t)$ evaluated from Eqs. (1)–(3) is plotted in Fig. 1a. The normalized component susceptibilities χ'/χ_0 and χ''/χ_0 versus $H_m/(J_c t)$ evaluated from Eqs. (4)–(6) are plotted in Fig. 1b.

It is observed from Fig. 1 that starting from a zero-field cooled state at $H = 0$, with increasing $H, -M$ increases in proportion to H and then approaches smoothly to saturation $-M_s$ and that with increasing H_m, χ' increases from $-\chi_0$ to zero, whereas χ'' increases to a maximum, χ''_m, at $H_m = H_m(\chi''_m)$ and then decreases to zero. All these are general features of CS magnetization and ac susceptibility.

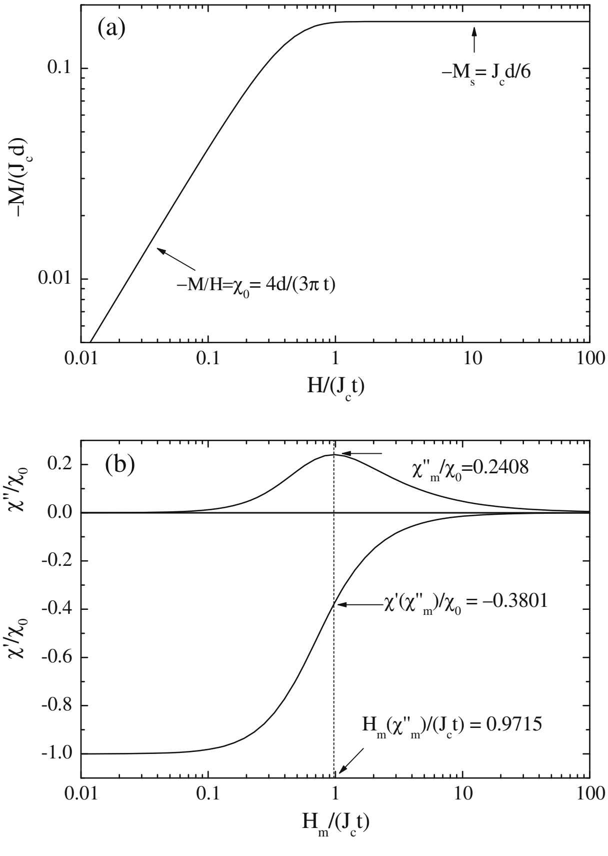

Fig. 1. Normalized initial magnetization curve (a) and normalized ac susceptibility as a function of the ac field amplitude and (b) analytically calculated for a thin disk of diameter d and thickness t with a constant J_c [9].

2.2. Scaling law for films

The analytic results for thin disks and for infinitely long thin strips, whose analytic results are given in [10,11], have been accurately evaluated in [6] as follows.

For the above thin disk with

$$\tau \equiv t/d, \tag{7}$$

$\tau\chi_0 = 4/(3\pi) = 0.4244$ and $H_m(\chi''_m) = 0.9715 J_c t$ lead to $H_m(\chi''_m) = -2.474 M_s/\chi_0$, where $M_s = -J_c d/6$ (see Fig. 1).

For the thin strip of thickness t and width w with

$$\tau \equiv t/w, \tag{8}$$

$\tau\chi_0 = \pi/4 = 0.7854$ and $H_m(\chi''_m) = 0.7844 J_c t$ lead to $H_m(\chi''_m) = -2.464 M_s/\chi_0$, where $M_s = -J_c w/4$.

Combining these results, we obtain a common scaling equation for both thin disk and thin strip geometries with an accuracy better than 0.25%:

$$H_m(\chi''_m) = -2.47\chi_0^{-1} M_s = 2.47 H_s, \tag{9}$$

where H_s is defined as $-M_s/\chi_0$ for convenience.

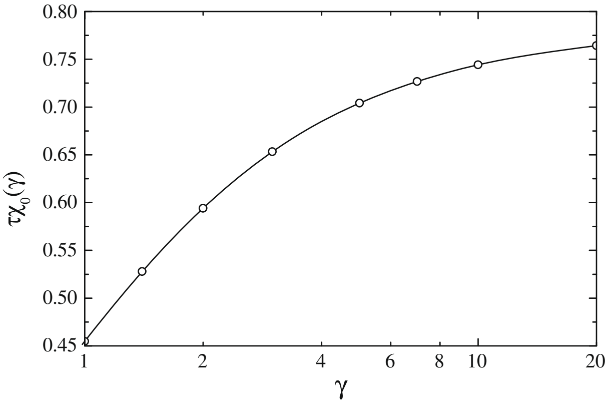

Fig. 2. $\tau\chi_0(\gamma)$ as a function of γ for rectangular films.