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Equivalent single-junction model for two-junction quantum interferometers with small inductance in the presence of noise

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ABSTRACT

We illustrate, in the limit of small values of the parameter β, a method to analyze the stochastic equations governing the time evolution of a two-junction quantum interferometer in the presence of noise. The analysis takes into account the different characteristic times of the two processes involved in the system, in such a way that the appropriate probability density function for each process is derived by means of one-dimensional Fokker–Planck equations. Therefore, the present analysis is equivalent to a generalization of the reduced single-junction model in the case of thermal fluctuations in the junctions. Voltage–current characteristics, transfer functions and circulating currents in 0-SQUIDs and π-SQUIDs are calculated by means of the proposed analytic approach.

1. Introduction

Ultra high sensitivity to magnetic field variation and high space resolution make Superconducting Quantum Interference Devices (SQUIDs) useful tools in experimental research [1–3]. Owing to their interesting dynamical properties, these devices are still being considered in the current literature. In fact, d.c. SQUIDs have been proposed as fundamental units in quantum computing [4–11]. These systems differ from a.c. SQUIDs, which consist of a superconducting loop interrupted by a single Josephson junction, by the presence of two junctions in the superconducting loop, in which a bias current I_B can be injected. An idealized representation of a d.c. SQUID is given by two point-like identical Josephson junctions placed symmetrically with respect to a circular superconducting loop at the midpoints of two equal current paths, as in Fig. 1. The currents I_1 and I_2 flow in the two loop branches when the current bias I_B and/or an external magnetic flux Φ_{ex} are applied to the system. In order to account for the magnetic energy of these currents, an inductance L is associated to each current branch. In the absence of noise and for negligible values of the capacitance parameter of the junctions, the RSJ model [1] suffices to describe the dynamical behaviour of the gauge-invariant superconducting phase differences φ_1 and φ_2 across the two Josephson junctions. Two coupled non-linear first-order ordinary differential equations can thus be analyzed to extract experimentally verifiable properties of the d.c. SQUID. An equivalent analysis can be performed for π-SQUIDs, where one of the two junctions in the device possesses an intrinsic π-shift in its superconducting phase difference [4,5]. We here recall that π-SQUIDs may be fabricated by exploiting solely the symmetry properties of d-wave superconductors [6,7] or by utilizing both s-wave and d-wave superconductors [8,9]. Lately, π-SQUIDs have been proposed as elementary memory cells in quantum computing [10–12].

When noise is considered, however, a stochastic term needs to be added to each of the two equations obtained by applying the RSJ model, so that the deterministic dynamical equations for a two-junction interferometer are rewritten as follows [1]:

$$\frac{\Phi_0}{2\pi R}\frac{d\varphi_1}{dt} + I_J \sin\varphi_1 = I_1 + I_{N1}(t), \tag{1a}$$

$$\frac{\Phi_0}{2\pi R}\frac{d\varphi_2}{dt} + I_J \sin\varphi_2 = I_2 + I_{N2}(t), \tag{1b}$$

where R and I_J are the resistive parameter and the maximum Josephson current of the two junctions, respectively, Φ_0 is the elementary flux quantum, and where φ_1 and φ_2 are now regarded as stochastic processes. The currents $I_{N1}(t)$ and $I_{N2}(t)$ represent two independent white noise terms such that

$$\langle I_{Ni}(t) I_{Nj(t')}\rangle = \frac{2k_BT}{R}\delta_{ij}\delta(t-t'), \qquad \text{for} \quad i,j = 1,2, \tag{2}$$

where the brackets stand for statistical average over noise. By considering the sets of Eqs. (1a) and (1b), Greenberg [13] and Chesca [14] have developed a detailed analytic description of the electrody-

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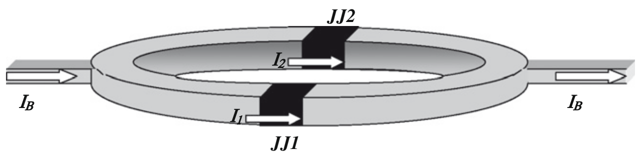

Fig. 1. Schematic representation of a two-junction quantum interferometer.

namic response of d.c. SQUID by means of a two-dimensional Fokker–Plank equation (FPE).

In the present work we point out that, in the limit of small values of the parameter $\beta = \frac{LI_J}{\Phi_0}$, besides the possibility of adopting a perturbation approach to the two-dimensional FPE, as done by Chesca [14], one can distinguish between two types of dynamical behaviors: one *fast*, the other *slow*, which would allow to treat the two equations independently. Indeed, by considering the slowly varying process, which appears as frozen in the dynamical equation for the *fast* process, we are able to apply a similar approach as in the Ambegaokar and Halperin [15] analysis of a single Josephson junction in the presence of noise. In this way, we may adopt a one-dimensional FPE in solving for the probability density function of the process. Therefore, by restricting the system studied by Grønbech-Jensen et al. [16] to overdamped junctions in the interferometer loop, in the present work we extend the analysis carried out in absence of noise by Romeo et al. [17] and by the authors themselves [18]. In the latter reports an effective single-junction model was adopted, for small values of β, by means of a perturbation analysis on the complete system of two coupled non-linear ordinary differential equations governing the dynamics of a d.c. SQUID.

The present work is thus organized as follows. In the next section the dynamical equations for the system are briefly recalled and the *fast* and *slow* processes are identified. In the third section, the Fokker–Planck equation for the *fast* process is solved for small values of the parameter β. The same is done for the *slow* process in sections four and five. Voltage – current characteristics and transfer functions are calculated in the sixth section. In the seventh section an important application of the present scheme is given in calculating the circulating currents characterizing the magnetic states of the system. Conclusions are drawn in the last section.

2. The dynamical equations

In order to briefly recall how the dynamical equations for the system are derived, let us first notice that the flux Φ threading the superconducting loop can be expressed as $\Phi = \Phi_{ex} + L(I_1 - I_2)$. Furthermore, notice that the quantization condition prescribes that $\frac{2\pi}{\Phi_0}\Phi + \varphi_1 - \varphi_2 = \pi N$, where N is an even integer for 0-SQUIDs and an odd integer for π-SQUIDs [19]. From this point on, we shall consider N even, so that $N = 2n$, n being an integer. In this way, Eqs. (1a) and (1b) can be rewritten as follows:

$$\frac{\Phi_0}{2\pi RI_J}\frac{d\varphi_1}{dt} = \frac{1}{2}\left(i_B - \frac{\psi_{ex} - n}{\beta}\right) - \sin\varphi_1 - \frac{\varphi_1 - \varphi_2}{4\pi\beta} + i_{N1}(t), \tag{3a}$$

$$\frac{\Phi_0}{2\pi RI_J}\frac{d\varphi_2}{dt} = \frac{1}{2}\left(i_B + \frac{\psi_{ex} - n}{\beta}\right) - \sin\varphi_2 + \frac{\varphi_1 - \varphi_2}{4\pi\beta} + i_{N2}(t), \tag{3b}$$

where $i_{Nk} = \frac{I_{Nk}}{I_{Ji}}$, $k = 1,2$ and $\psi_{ex} = \frac{\Phi_{ex}}{\Phi_0}$. Naturally, the above equations can be readily written also for π-SQUIDs by substituting n with $\frac{2n-1}{2}$. Recall that n is the number of initially trapped fluxons in the superconducting loop.

By introducing the new processes $x_1 = \frac{\varphi_1 + \varphi_2}{2}$ and $x_2 = \frac{\varphi_1 - \varphi_2}{2} = \pi\left(n - \frac{\Phi}{\Phi_0}\right)$, we can rewrite Eqs. (3a) and (3b) in the following form:

$$\frac{\Phi_0}{2\pi RI_J}\frac{dx_1}{dt} = \frac{i_B}{2} - \cos x_2 \sin x_1 + \frac{1}{2}(i_{N1}(t) + i_{N2}(t)), \tag{4a}$$

$$\frac{\Phi_0}{2\pi RI_J}\frac{dx_2}{dt} = -\sin x_2 \cos x_1 - \frac{x_2}{2\pi\beta} + \frac{n - \psi_{ex}}{2\beta} + \frac{1}{2}(i_{N1}(t) - i_{N2}(t)). \tag{4b}$$

Notice that the quantities $i_{N\pm}(t) = \left(\frac{i_{N1}(t) \pm i_{N2}(t)}{2}\right)$ are still two independent white noise terms. Let us now introduce the normalized effective potential for the d.c. SQUID [20]:

$$W(x_1, x_2) = (1 - \cos x_2 \cos x_1) - \frac{i_B}{2}x_1 + \frac{1}{4\pi\beta}(x_2 + \pi\psi_{ex} - n\pi)^2. \tag{5}$$

The time-evolution of the processes x_1 and x_2, by introducing the normalized time $\tau = \frac{2\pi RI_J}{\Phi_0}t$, can be obtained by setting:

$$\frac{dx_1}{d\tau} = -\frac{\partial W}{\partial x_1} + i_{N+}(\tau), \tag{6a}$$

$$\frac{dx_2}{d\tau} = -\frac{\partial W}{\partial x_2} + i_{N-}(\tau), \tag{6b}$$

where the noise terms $i_{N\pm}(\tau)$ now satisfy the following relations:

$$\langle i_{N+}(\tau) i_{N+}(\tau')\rangle = \frac{k_B T}{E_J}\delta(\tau - \tau') = \Gamma\delta(\tau - \tau'), \tag{7a}$$

$$\langle i_{N-}(\tau) i_{N-}(\tau')\rangle = \frac{k_B T}{E_J}\delta(\tau - \tau') = \Gamma\delta(\tau - \tau'), \tag{7b}$$

where $E_J = \frac{I_J\Phi_0}{2\pi}$ and $\Gamma = \frac{k_B T}{E_J}$ and the brackets denote expectation values.

To the stochastic Eqs. (6a) and (6b) the following Fokker–Planck equation (FPE) for the probability density $p(\mathbf{x}, \tau)$ is associated

$$\frac{\partial p(\mathbf{x}, \tau)}{\partial \tau} = \nabla \cdot (p(\mathbf{x}, \tau)\nabla W) + \frac{\Gamma}{2}\nabla^2 p(\mathbf{x}, \tau), \tag{8}$$

where $\mathbf{x} \equiv (x_1, x_2)$ and the dot represents the usual inner product between vectors. The density function $p(\mathbf{x}, \tau)$ is subject to the normalization condition

$$\int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{+\infty} dx_2 p(\mathbf{x}, \tau) = 1. \tag{9}$$

As specified before, this problem has already been thoroughly treated by Greenberg [13] and Chesca [14]. In this paper, however, we consider again the dynamical structure of (4a) and (4b) to show that an effective single-junction dynamical model for the system, proposed in Refs. [17,18], is still valid in the presence of noise, as also reported by Grønbech-Jensen et al. [16].

We start by stating that, with the aid of Ref. [21], one can establish existence of the stationary regime for the solution of FPE (8), as reported in the Appendix A. Therefore, a unique stationary density function $p(x_1, x_2)$ which does not depend on the normalized time τ can be determined by the following equation:

$$\mathbf{L}p(\mathbf{x}) = \nabla \cdot (p(\mathbf{x})\nabla W) + \frac{\Gamma}{2}\nabla^2 p(\mathbf{x}) = 0. \tag{11}$$

The above equation can be recast in a more convenient form, by noticing that $\mathbf{L}p(\mathbf{x}) = \mathbf{L}_1 p(\mathbf{x}) + \mathbf{L}_2 p(\mathbf{x})$, where

$$\mathbf{L}_k p(\mathbf{x}) = \frac{\partial}{\partial x_k}\left(p(\mathbf{x})\frac{\partial W}{\partial x_k}\right) + \frac{\Gamma}{2}\frac{\partial^2}{\partial x_k^2}p(\mathbf{x}) = 0, \text{ for } k = 1, 2. \tag{12}$$

This allows us to write the density function as follows

$$p(x_1, x_2) = q(x_1)p_2(x_1, x_2), \tag{13}$$

and to consider, for the particular role played by the constant β in the dynamical Eq. (4b) and in the stationary FPE (11), the process x_2 as a *fast* process. Indeed, from a strictly physical point of view, we notice that a different characteristic time can be introduced in