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On Ruby's solid angle formula and some of its generalizations

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ABSTRACT

Using the Mellin–Barnes representation, we show that Ruby's solid angle formula and some of its generalizations may be expressed in a compact way in terms of the Appell F_4 and Lauricella F_C functions.

1. Introduction

Ruby's formula, giving the solid angle subtended at a disk source by a coaxial parallel-disk detector [1], is the following:

$$G=\frac{R_D}{R_S}\int_0^\infty dk\frac{e^{-kd}}{k}J_1(kR_S)J_1(kR_D), \tag{1}$$

where R_S and R_D are respectively the radius of the source and of the detector, d is the distance between the source and the detector and $J_1(x)$ is the Bessel function of first kind and order 1.

Until [2], where an expression in terms of complete and incomplete elliptic integrals has been given, it seems that Ruby's formula had not been expressed in a closed form. A double series representation had been previously mentioned in [3] but it was concluded in [4] that the convergence region of this double series is restricted in a way that when the detector is too close to the source, one had to compute the integral by means of numerical methods.

In the next section, we will see that with the help of the Mellin–Barnes (MB) representation method (see e.g. [5] for an introduction), Ruby's formula may be expressed in a compact way in terms of the Appell function F_4.

Generalizations of Ruby's formula, which have been treated mainly in [2] but not always obtained in closed form, will also be considered within the same approach, in a subsequent section where their expressions in terms of the Lauricella function F_C will be given.

2. Ruby's formula

The Bessel function $J_1(z)$ has the following MB representation, valid for $z>0$, see [5]:

$$J_1(z)=\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} ds\left(\frac{z}{2}\right)^{1-2s}\frac{\Gamma(s)}{\Gamma(2-s)}, \tag{2}$$

where, for absolute convergence of the integral, the constant c, which is the real part of s (since the chosen integration path is a vertical line in the s-complex plane) has to belong to the interval $]0,\frac{1}{2}[$ [5].

Inserting twice this integral representation in Eq. (1) we get

$$G=\frac{R_D^2}{4}\left(\frac{1}{2i\pi}\right)^2\int_{c-i\infty}^{c+i\infty}ds\int_{c'-i\infty}^{c'+i\infty}dt\left(\frac{R_S}{2}\right)^{-2s}\left(\frac{R_D}{2}\right)^{-2t}\frac{\Gamma(s)}{\Gamma(2-s)}\frac{\Gamma(t)}{\Gamma(2-t)}$$
$$\times\int_0^\infty dk\, k^{1-2s-2t}e^{-kd}, \tag{3}$$

where $c=\Re(s)\in]0,\frac{1}{2}[$ and $c'=\Re(t)\in]0,\frac{1}{2}[$.

Performing the k-integral leads to the following 2-fold MB representation:

$$G=\left(\frac{1}{2i\pi}\right)^2\left(\frac{R_D}{2d}\right)^2$$
$$\times\int_{c-i\infty}^{c+i\infty}ds\int_{c'-i\infty}^{c'+i\infty}dt\left(\frac{R_S}{2d}\right)^{-2s}\left(\frac{R_D}{2d}\right)^{-2t}\frac{\Gamma(s)\Gamma(t)\Gamma(2-2s-2t)}{\Gamma(2-s)\Gamma(2-t)}, \tag{4}$$

with the constraint $\Re(s+t)<1$, which is fulfilled.

To compute this integral one can directly apply the general method described in [6]. If one follows this approach, one will find three double series representations (one of them being the one

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mentioned in [3] and studied in [4]), converging in three different regions of values of the parameters R_S, R_D and d.

It is however even simpler to notice that, by using the duplication formula for the Euler gamma function

$$\Gamma(2s)=\frac{1}{\sqrt{\pi}}2^{2s-1}\Gamma(s)\Gamma\left(s+\frac{1}{2}\right), \tag{5}$$

one has

$$G=\frac{2}{\sqrt{\pi}}\left(\frac{R_D}{2d}\right)^2\left(\frac{1}{2i\pi}\right)^2\int_{c-i\infty}^{c+i\infty}ds\int_{c'-i\infty}^{c'+i\infty}dt\left(\frac{R_S}{d}\right)^{-2s}\left(\frac{R_D}{d}\right)^{-2t}$$
$$\times\frac{\Gamma(s)\Gamma(t)}{\Gamma(2-s)\Gamma(2-t)}\Gamma(1-s-t)\Gamma\left(\frac{3}{2}-s-t\right), \tag{6}$$

which, since $\Gamma(\frac{3}{2})=2/\sqrt{\pi}$, is nothing but the MB representation of the Appell F_4 function [7]

$$G=\left(\frac{R_D}{2d}\right)^2F_4\left(1,\frac{3}{2},2,2;-\left(\frac{R_S}{d}\right)^2,-\left(\frac{R_D}{d}\right)^2\right) \tag{7}$$

from where, by definition, one gets the double series representation:

$$G=\left(\frac{R_D}{2d}\right)^2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}}{m!n!}\frac{(1)_{m+n}(\frac{3}{2})_{m+n}}{(2)_m(2)_n}\left(\frac{R_S}{d}\right)^{2m}\left(\frac{R_D}{d}\right)^{2n}, \tag{8}$$

where $(a)_m=\Gamma(a+m)/\Gamma(a)$ is the Pochhammer symbol.

The series (8) is the same as the one studied in [4] with convergence region given[2] by $R_S+R_D<d$. This confirms the analysis performed in [4].

The well-known analytic continuation formula [7]

$$F_4(a,b,c,d;x,y)=\frac{\Gamma(d)\Gamma(b-a)}{\Gamma(d-a)\Gamma(b)}(-y)^{-a}F_4\left(a,a+1-d,c,a+1-b;\frac{x}{y},\frac{1}{y}\right)$$
$$+\frac{\Gamma(d)\Gamma(a-b)}{\Gamma(d-b)\Gamma(a)}(-y)^{-b}F_4\left(b+1-d,b,c,b+1-a;\frac{x}{y},\frac{1}{y}\right) \tag{9}$$

and the symmetric relation obtained by exchanging x with y and c with d allow us to obtain double series representations valid in the regions $R_S+d<R_D$ and $R_D+d<R_S$.

Series representations valid in other ranges of values of the parameters than those given above may be found in [8], where a full analytic continuation study has been performed.

3. Generalization

The same technique may be used to compute the more general integral:

$$I_{(l,m_1,\ldots,m_N)}=\int_0^\infty dk\,k^l\,e^{-kd}\prod_{j=1}^{N}J_{m_j}(kR_j), \tag{10}$$

where l and the m_j are such that the integral converges, and $R_j>0$ for all $j\in\{1,\ldots,N\}$.

In this case, we use the following MB representation for the Bessel functions (valid for $z>0$):

$$J_m(z)=\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}ds\left(\frac{z}{2}\right)^{m-2s}\frac{\Gamma(s)}{\Gamma(1+m-s)}, \tag{11}$$

where $c=\Re(s)\in\,]0,\Re(m)/2[$.

Notice that this integral is defined only when $m>0$ but we will see that one can relax this constraint at the end of the calculations by appealing to analytic continuation.

[2] The convergence regions of the double series representations of the Appell functions are well-known: it is straightforward to get them by Horn's method, see for instance [6] or [7].

Inserting Eq. (11) in Eq. (10) we get

$$I_{(l,m_1,\ldots,m_N)}=\prod_{j=1}^{N}\left[\frac{1}{2i\pi}\int_{c_j-i\infty}^{c_j+i\infty}ds_j\left(\frac{R_j}{2}\right)^{m_j-2s_j}\frac{\Gamma(s_j)}{\Gamma(1+m_j-s_j)}\right]$$
$$\times\int_0^\infty dk\,e^{-kd}k^{l+\sum_{j=1}^{N}(m_j-2s_j)}. \tag{12}$$

The k-integral gives $d^{-1-l-\sum_{j=1}^{N}(m_j-2s_j)}\Gamma(1+l+\sum_{j=1}^{N}(m_j-2s_j))$ with the constraint $\Re(1+l+\sum_{j=1}^{N}(m_j-2s_j))>0$. Let us suppose that this constraint is fulfilled (in all particular cases considered in [2], it is always possible to satisfy this constraint by an appropriate choice of the c_j).

Then, applying Eq. (5), one may conclude that

$$I_{(l,m_1,\ldots,m_N)}=\frac{1}{\sqrt{\pi}}\left(\frac{2}{d}\right)^l\frac{1}{d}\prod_{j=1}^{N}\left[\left(\frac{R_j}{d}\right)^{m_j}\frac{1}{2i\pi}\int_{c_j-i\infty}^{c_j+i\infty}ds_j\left(\frac{R_j}{d}\right)^{-2s_j}\frac{\Gamma(s_j)}{\Gamma(1+m_j-s_j)}\right]$$
$$\times\Gamma\left(\sum_{j=1}^{N}\left(\frac{m_j}{2}-s_j\right)+\frac{l+1}{2}\right)\Gamma\left(\sum_{j=1}^{N}\left(\frac{m_j}{2}-s_j\right)+\frac{l}{2}+1\right). \tag{13}$$

As a particular check, it is easy to derive Eq. (6) from Eq. (13) by putting $N=2$, $m_1=m_2=1$, $R_1=R_D$, $R_2=R_S$ and $l=-1$, and multiplying by R_D/R_S.

In the case where $\sum_{j=1}^{N}m_j/2+(l+1)/2$ and $\sum_{j=1}^{N}m_j/2+l/2+1$ are positive numbers, we recognize in Eq. (13) the MB representation of the multiple Lauricella function $F_C^{(N)}$ (modulo an overall factor) [9].

We therefore have

$$I_{(l,m_1,\ldots,m_N)}=\frac{1}{\sqrt{\pi}}\left(\frac{2}{d}\right)^l\frac{1}{d}\frac{\Gamma\left(\sum_{j=1}^{N}\frac{m_j}{2}+\frac{l+1}{2}\right)\Gamma\left(\sum_{j=1}^{N}\frac{m_j}{2}+\frac{l}{2}+1\right)}{\prod_{j=1}^{N}\Gamma(1+m_j)}\prod_{j=1}^{N}\left(\frac{R_j}{d}\right)^{m_j}$$
$$\times F_C^{(N)}\left(\sum_{j=1}^{N}\frac{m_j}{2}+\frac{l+1}{2},\sum_{j=1}^{N}\frac{m_j}{2}+\frac{l}{2}+1,1+m_1,\ldots,1+m_N;-\frac{R_1^2}{d^2},\ldots,-\frac{R_N^2}{d^2}\right). \tag{14}$$

Lauricella functions are the generalizations of Appell functions and $F_C^{(2)}$ is, obviously, nothing but the Appell F_4 function.

The integral representation in Eq. (13) has been obtained with the initial constraint that the m_j and R_j are strictly positive for all $j\in\{1,\ldots,N\}$. By analytic continuation, it is however possible to include other values for these parameters (and, among others, the important case where some of the m_j are equal to zero) since Eq. (13) is defined as long as the quantities $\sum_{j=1}^{N}m_j/2+(l+1)/2$ and $\sum_{j=1}^{N}m_j/2+l/2+1$ are not negative integers. In fact, in all particular situations considered in [2] these two quantities are positive numbers. Therefore one may directly use Eq. (14) to compute them and this is done in a subsection to follow.

Using the multiple series representation of the multiple Lauricella function $F_C^{(N)}$ [9], one obtains:

$$I_{(l,m_1,\ldots,m_N)}=\frac{1}{\sqrt{\pi}}\left(\frac{2}{d}\right)^l\frac{1}{d}\frac{\Gamma\left(\sum_{j=1}^{N}\frac{m_j}{2}+\frac{1+l}{2}\right)\Gamma\left(\sum_{j=1}^{N}\frac{m_j}{2}+\frac{l}{2}+1\right)}{\prod_{j=1}^{N}\Gamma(1+m_j)}\prod_{j=1}^{N}\left(\frac{R_j}{d}\right)^{m_j}$$
$$\times\sum_{k_1=0}^{\infty}\cdots\sum_{k_N=0}^{\infty}\frac{\left(\sum_{j=1}^{N}\frac{m_j}{2}+\frac{l+1}{2}\right)_{\sum_{j=1}^{N}k_j}\left(\sum_{j=1}^{N}\frac{m_j}{2}+\frac{l}{2}+1\right)_{\sum_{j=1}^{N}k_j}}{\prod_{j=1}^{N}((1+m_j)_{k_j}k_j!)}$$
$$\times(-1)^{\sum_{j=1}^{N}k_j}\left(\frac{R_1}{d}\right)^{2k_1}\cdots\left(\frac{R_N}{d}\right)^{2k_N} \tag{15}$$

where as before $(a)_m$ is the Pochhammer symbol.

This multiple series converges in the region $\sum_{j=1}^{N}|R_j|<|d|$ [9].