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Correlated statistical uncertainties in coded-aperture imaging

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ABSTRACT

In nuclear security applications, coded-aperture imagers can provide a wealth of information regarding the attributes of both the radioactive and nonradioactive components of the objects being imaged. However, for optimum benefit to the community, spatial attributes need to be determined in a quantitative and statistically meaningful manner. To address a deficiency of quantifiable errors in coded-aperture imaging, we present uncertainty matrices containing covariance terms between image pixels for MURA mask patterns. We calculated these correlated uncertainties as functions of variation in mask rank, mask pattern over-sampling, and whether or not anti-mask data are included. Utilizing simulated point source data, we found that correlations arose when two or more image pixels were summed. Furthermore, we found that the presence of correlations was heightened by the process of over-sampling, while correlations were suppressed by the inclusion of anti-mask data and with increased mask rank. As an application of this result, we explored how statistics-based alarming is impacted in a radiological search scenario.

1. Scope, motivation and focus

Originally designed as a method to observe high-energy photons in astronomical applications [1,2], coded-aperture imaging is a mature, indirect technique for obtaining spatial information in a variety of imaging applications [3,4]. As an elaborate extrapolation from a single pinhole aperture, a mask with structured pattern is placed between a radiation source and a position-sensitive detector, resulting in a modulated hit pattern on the detector [1–4]. The resulting hit pattern is computationally decoded to produce an image of radiation emanating from the field-of-view.

Coded apertures are currently utilized across a spectrum of imaging applications, including astrophysics [5,6] and medicine [7–9]. The nuclear security community has successfully applied coded-aperture techniques to gamma-ray imaging of radioisotopes for arms control and nuclear nonproliferation applications [10]. In certain cases, the coded-aperture technique provides simultaneously collected spectral information in addition to the spatial location of the sources in question [4,11]. Both of these capabilities are valuable to the nuclear community since they provide access to the shape, size, isotopic composition, and activity of the radioisotopes of interest.

In order to be of optimum benefit to the community, quantitative results that rely on statistically meaningful analyses of imaged data are needed. However, coded-aperture imaging was designed for point source imaging, and for cases where sources occupy a single pixel, uncertainties are well-known. Coded-aperture imaging can also be used for extended sources. For point sources that contain strength in adjacent image pixels or for extended sources that cover multiple image pixels, a measure of the total strength of a source requires summing multiple image pixels together. If correlations between image pixels exist, then covariance terms must be properly included in the uncertainty calculation.

The focus of the current study is on the calculation of the uncertainties for the sum of multiple image pixels and these uncertainties as a function of a few important imaging variables. Unlike previous studies, we do not presuppose the absence of correlations when summing multiple image pixels during specific calculations of total source uncertainty. The article maintains the following outline: Section 2 introduces the imaging variables manipulated in the current study. In Section 3, mathematical formalism of the covariance problem is developed, while Section 4 examines image pixel values for degrees of correlation. Section 5 presents the degree of correlation as a function of the three image variables, and Section 6 discusses applications related to the detection significance within the nuclear security sector.

2. Coded-aperture imaging variables of interest

The imaging variables adjusted in the current work are mask rank, whether or not anti-mask data are included, and mask-pattern oversampling. Each of these variables is important for

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coded-aperture imaging performed with MURA mask, and may not be applicable to other mask types. A modified uniformly redundant array (MURA) is a common mask pattern yielding high signal-to-noise images [12]. They consist of an equal number of open and closed mask elements, giving an open fraction of about 50%.

The rank of the mask, R, refers to the prime number on which the MURA pattern is based. Even though other mask pattern arrangements have been found to equal and/or surpass MURA signal-to-noise ratios [13,8,14,15], MURA patterns exhibit ideal imaging properties because the number of artifacts is minimized due to auto-correlative properties [16]. In addition, due to the anti-symmetry of the MURA pattern under rotation, background subtraction is uncomplicated and efficient [17].

The accumulation of anti-mask detector counts afforded by MURA patterns provides a significant advantage for image quality [10]. In rotating an anti-symmetric mask by 90° and exposing the detector plane, an anti-mask exposure is obtained because the spatial locations of the open and closed mask positions are exchanged [17,18]. Subtracting the anti-mask counts from the mask counts suppresses image artifacts and provides an *in situ* background subtraction [11].

The concept of over-sampling relates to the number of detector pixels that sample each mask opening. In a one-to-one mapping between the detector plane and mask, each opening or closure in the mask is resolved by one detector pixel. This relationship between mask and detector is referred to as single-sampling. When an image is double-sampled, the number of detector pixels covering each mask opening is increased by 2^2 (two times increase in each dimension). The process of over-sampling serves two purposes related to image quality. First, over-sampling aids in the aesthetic problem of pixelation. For detector planes and instrumentation with an inherently large pixel size, the fourfold increase in the number of image pixels increases the spatial sampling, allowing the shape of objects in the field-of-view to be more easily observed. Second, the number of effective lines-of-sight forming the image increases, which adds to the information present when reconstructing. By remembering that the mask is an effective conglomeration of individual pinholes, each pinhole forms an unique response on the detector (i.e., a basis vector). When the detector plane is over-sampled, there is an increase in the number of basis vectors available to reconstruct the image. However, these basis vectors are not completely independent; rather, they are linear combinations of the original, single-sampled, orthogonal basis vectors.

3. Correlation and covariance within coded-aperture imaging

Since coded-aperture image reconstruction involves a cross-correlation of the detector hit pattern with the MURA mask pattern [3,12], the value of each reconstructed image pixel (I_n) can be expressed as

$$I_n = \sum_i M_{n,i} D_i, \quad (1)$$

where D_i represents the measured detector counts in pixel i, and $M_{n,i}$ is the transfer (or mask) function which is based on the MURA mask pattern with a value of $+1$ for open mask elements and -1 for closed mask elements (with the exception of the central mask element, which is closed but always set to $+1$ in order to achieve a delta function response for a point source [19]). Note that in our notation, a single index is used for an image pixel; it should be understood that two-dimensional image pixel indices are collapsed into one index for clarity in the expressions that follow. If the *detector* pixels (D_i) are assumed to have independent errors, i.e., no covariance ($\sigma^2_{D_iD_j} = 0$, for $i \neq j$), the uncertainty on any single image pixel follows as

$$\sigma^2_{I_n} = \sum_i \left(\frac{\partial I_n}{\partial D_i}\right)^2 \sigma^2_{D_i}, \quad (2)$$

where the partial derivatives are given by the decoding array sampled for a particular image pixel:

$$\left(\frac{\partial I_n}{\partial D_i}\right) = M_{n,i} = \pm 1.$$

Also since the associated errors for the detector pixels are determined by Poisson statistics (i.e., $\sigma_{D_i} \approx \sqrt{D_i}$):

$$\sum_i \sigma^2_{D_i} = N, \quad (3)$$

where N is simply the total number of counts in the detector from both source and background. Therefore, the uncertainty on any single-image pixel simplifies to

$$\sigma^2_{I_n} = N, \quad (4)$$

i.e., the variance. When including anti-mask data (see Section 4.1), the total error for single-image pixels will take the form:

$$\sigma^2_{I_n} = \sum_i \sigma^2_{D_i} = \sum_i \left[\left(\sigma^M_{D_i}\right)^2 + \left(\sigma^A_{D_i}\right)^2\right] = N^M + N^A, \quad (5)$$

where the superscripts "M" and "A" represent the detector counts in a given mask or anti-mask exposure, respectively.

At this point, we may be tempted to naively assume that the total error for the sum of two image pixels is

$$\sigma^2_{I_m + I_n} = \sigma^2_{I_m} + \sigma^2_{I_n} = 2N. \quad (6)$$

However Eq. (6) assumes an independent nature for the image pixel uncertainties and the absence of any correlation between them. Until this assumption can be confirmed for multiple image pixels, we must calculate the covariances between image pixels, namely, $\sigma^2_{I_mI_n}$ [20]. Note that in Eq. (1) the vector containing detector counts is the same for each image pixel calculation. The reapplication of the entire dataset to calculate each image pixel value ensures that the image pixel uncertainties will be correlated, which points to the need for the use of the full covariance matrix. Therefore, if two image pixels are summed, the uncertainty for the sum is

$$\sigma^2_{I_m + I_n} = \sigma^2_{I_m} + \sigma^2_{I_n} + 2\sigma^2_{I_mI_n}. \quad (7)$$

Explicitly, the covariance term in Eq. (7) takes the following form:

$$\sigma^2_{I_mI_n} = \sum_i \left[\left(\frac{\partial I_m}{\partial D_i}\right)\left(\frac{\partial I_n}{\partial D_i}\right)\sigma^2_{D_i}\right] + \sum_i \sum_{j \neq i} \left[\left(\frac{\partial I_m}{\partial D_i}\right)\left(\frac{\partial I_n}{\partial D_j}\right)\sigma^2_{D_iD_j}\right]. \quad (8)$$

Because we are assuming that the detector pixel uncertainties are uncorrelated ($\sigma^2_{D_iD_j} = 0$), the second term in Eq. (8) goes to zero and the uncertainty for the sum of two image pixels becomes

$$\sigma^2_{I_m + I_n} = \sum_i \left[\left(\frac{\partial I_m}{\partial D_i}\right)^2 \sigma^2_{D_i} + \left(\frac{\partial I_n}{\partial D_i}\right)^2 \sigma^2_{D_i} + 2\left(\frac{\partial I_m}{\partial D_i}\right)\left(\frac{\partial I_n}{\partial D_i}\right)\sigma^2_{D_i}\right], \quad (9)$$

where the first two terms are instances of Eq. (2), and the third term is the remainder of Eq. (8). Having established the values of the partial derivatives as decoder elements of ± 1 and knowing the detector pixel counts, the uncertainties for image pixel sums are calculable. Further, it is clear from Eq. (9) that the first two partial derivative coefficients are squared, giving always $+1$, while the partials in front of the third term could be ± 1. To verify the mathematical result, an examination of simulated coded-aperture images is now presented.