Nuclear Instruments and Methods in Physics Research A 600 (2009) 471–477

Contents lists available at ScienceDirect

Nuclear Instruments and Methods in Physics Research A

journal homepage: www.elsevier.com/locate/nima

The global covariance matrix of tracks fitted with a Kalman filter and an application in detector alignment

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ARTICLE INFO

Article history:
Received 13 October 2008
Received in revised form
15 November 2008
Accepted 18 November 2008
Available online 3 December 2008

Keywords:
Alignment of tracking detectors
Kalman filter

ABSTRACT

We present an expression for the covariance matrix of the set of state vectors describing a track fitted with a Kalman filter. We demonstrate that this expression facilitates the use of a Kalman filter track model in a minimum χ^2 algorithm for the alignment of tracking detectors. We also show that it allows to incorporate vertex constraints in such a procedure without refitting the tracks.

1. Introduction

Minimum χ^2 algorithms for the alignment of tracking detectors generally come in two flavors, namely those that ignore and those that do not ignore the correlations between hit residuals. The former are sometimes called *local* or *iterative* methods while the latter are called *global* or *closed-form* methods [1]. The advantage of the closed-form methods is that for an alignment problem in which the measurement model is a linear function of both track and alignment parameters the solution that minimizes the total χ^2 can be obtained with a single pass over the data.

The covariance matrix for the track parameters is an essential ingredient to the closed-form alignment approach [2]. If the track fit is performed using the standard expression for the least squares estimator (sometimes called the *standard* or *global* fit method), the computation of the covariance matrix is a natural part of the track fit. This is why previously reported implementations of the closed-form alignment procedure (e.g. Refs. [3–8]) make use of the standard fit.

In contrast most modern particle physics experiments rely on a Kalman filter track fit [9,10] for default track reconstruction. The Kalman filter is less computationally expensive than the standard fit and facilitates an easy treatment of multiple scattering in the form of process noise. However, the computation of the covariance matrix in the common Kalman track fit is not complete: the correlations between track parameters at different position along the track are not calculated. In the presence of process noise these correlations are non-trivial. Consequently, the result of the common Kalman track fit cannot be used directly in a closed-form alignment procedure.

In this paper, we present the expressions for the computation of the global covariance matrix—the covariance matrix for all parameters in the track model—in a Kalman filter track fit. We show how this result can be used in an alignment procedure. Furthermore, using similar expressions we demonstrate how vertex constraints can be applied in the alignment without refitting the tracks in the vertex. To illustrate that our approach leads to a functional closed-form alignment algorithm, we present some results obtained for the alignment of the LHCb vertex detector with Monte Carlo simulated data.

An important motivation for extending the Kalman track fit for use in a closed-form alignment approach is that the estimation of alignment parameters is not independent of the track model. Typically, in closed-form alignment procedures the track model used in the alignment is different from that used in the track reconstruction for physics analysis, which in practise is always a Kalman filter. Sometimes the track model in the alignment is simplified, ignoring multiple scattering corrections or the magnetic field. The imperfections in the track model used for alignment will partially be absorbed in calibration parameters. Consequently, in order the guarantee consistency between track model and detector alignment, it is desirable to use the default track fit in the alignment procedure.

The Kalman filter has also been proposed for the estimation of the alignment parameters themselves [11]. This method for alignment is an alternative formulation of the closed-form alignment approach that is particularly attractive if the number of alignment parameter is large. Our results for the global

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0168-9002/$ - see front matter
doi:10.1016/j.nima.2008.11.094

covariance matrix of the Kalman filter track model and for vertex constraints can eventually be applied in such a Kalman filter alignment procedure.

2. Minimum χ^2 formalism for alignment

To show that the global covariance matrix of the track parameters is an essential ingredient to the closed-form alignment approach, we briefly revisit the minimum χ^2 formalism for alignment. Consider a track χ^2 defined as

$$\chi^2 = [m - h(x)]^{\mathrm{T}} V^{-1} [m - h(x)] \tag{1}$$

where m is a vector of measured coordinates, V is a (usually diagonal) covariance matrix, $h(x)$ is the measurement model and x is the vector of track parameters. Note that Eq. (1) is a matrix expression: m and h are vectors and V is a symmetric matrix, all with dimension equal to the number of measurements.

For a linear expansion of the measurement model around an initial estimate x_0 of the track parameters

$$h(x) = h(x_0) + H(x - x_0)$$

where

$$H = \left.\frac{\partial h(x)}{\partial x}\right|_{x_0}$$

is sometimes called the derivative or projection matrix, the condition that the χ^2 be minimal with respect to x can be written as

$$0 \equiv \frac{\mathrm{d}\chi^2}{\mathrm{d}x} = -2H^{\mathrm{T}} V^{-1} [m - h(x_0) - H(x - x_0)].$$

The solution to this system of equations is given by the well known expression for the least squares estimator

$$x = x_0 - CH^{\mathrm{T}} V^{-1} [m - h(x_0)] \tag{2}$$

where the matrix C is the covariance matrix for x

$$C = (H^{\mathrm{T}} V^{-1} H)^{-1}. \tag{3}$$

If the measurement model is not linear, i.e. if H depends on x, expression Eq. (2) can be applied iteratively, until a certain convergence criterion is met, for example defined by a minimum change in the χ^2. In that case it makes sense to write Eq. (2) in terms of the first and second derivative of the χ^2 at the current estimate x_0

$$x - x_0 = -\left(\left.\frac{\mathrm{d}^2\chi^2}{\mathrm{d}x^2}\right|_{x_0}\right)^{-1} \left.\frac{\mathrm{d}\chi^2}{\mathrm{d}x}\right|_{x_0}$$

and regard the iterative minimization procedure as an application of the Newton–Raphson method.

We now consider an extension of the measurement model with a set of calibration parameters α

$$h(x) \rightarrow h(x, \alpha).$$

The parameters α are considered common to all tracks in a particular calibration sample. We estimate α by minimizing the sum of the χ^2 values of the tracks simultaneously with respect to α and the track parameters x_i of each track i

$$\frac{\partial \sum_i \chi_i^2}{\partial \alpha} = 0 \quad \text{and} \quad \forall_i \frac{\partial \chi_i^2}{\partial x_i} = 0. \tag{4}$$

Note that the index i refers to the track and not to a component of the vector x. We will omit the index from now on and consider only the χ^2 contribution from a single track.

The number of parameters in the minimization problem above scales with the number of tracks. If the number of tracks is large enough, a computation that uses an expression for the least squares estimator analogous to Eq. (2) is computationally too expensive. A more practical method relies on a computation in two steps. First, track parameters are estimated for an initial set of calibration parameters α_0. Subsequently, the total χ^2 is minimized with respect to α taking into account the dependence of x_i on α, e.g. through the total derivative

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} = \frac{\partial}{\partial \alpha} + \frac{\mathrm{d}x}{\mathrm{d}\alpha}\frac{\partial}{\partial x}. \tag{5}$$

The derivative matrix $\mathrm{d}x/\mathrm{d}\alpha$ in Eq. (5) follows from the condition that the χ^2 of the track remains minimal with respect to x, which can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\partial \chi^2}{\partial x} = 0$$

and results in

$$\frac{\mathrm{d}x}{\mathrm{d}\alpha} = -\frac{\partial^2\chi^2}{\partial\alpha\,\partial x}\left(\frac{\partial^2\chi^2}{\partial x^2}\right)^{-1}. \tag{6}$$

Note that if the problem is linear this derivative is independent of the actual value of x or α. Consequently, in this limit this expression remains valid even if the track χ^2 was not yet minimized with respect to x.

The condition that the total χ^2 of a sample of tracks be minimal with respect to both track and alignment parameters can now be expressed as

$$0 \equiv \frac{\mathrm{d}\chi^2}{\mathrm{d}\alpha}. \tag{7}$$

For M alignment parameter this defines a system of M coupled non-linear equations. In analogy with the procedure introduced for the track χ^2 minimization above we search for a solution by linearizing the minimum χ^2 condition around an initial value α_0 and solving the linear system of M equations

$$\left.\frac{\mathrm{d}^2\chi^2}{\mathrm{d}\alpha^2}\right|_{\alpha_0} \Delta\alpha = -\left.\frac{\mathrm{d}\chi^2}{\mathrm{d}\alpha}\right|_{\alpha_0} \tag{8}$$

for $\Delta\alpha$. In the remainder of this section we derive the expressions for these derivatives.

To simplify the notation we define the residual vector of the track

$$r = m - h(x, \alpha)$$

and its derivative to α

$$A_{k\ell} \equiv \frac{\partial r_k}{\partial \alpha_\ell}.$$

We linearize r around the expansion point $(x(\alpha_0), \alpha_0)$, and using Eq. (6) obtain for any total derivative to α

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} = \frac{\partial}{\partial\alpha} - A^{\mathrm{T}} V^{-1} HC \frac{\partial}{\partial x}.$$

(The minus sign appears because H is the derivative of h and not of r.) In this expression we have substituted the covariance matrix for C for x. The first and second derivatives of the χ^2 contribution of a single track are now given by

$$\frac{\mathrm{d}\chi^2}{\mathrm{d}\alpha} = 2A^{\mathrm{T}} V^{-1}\left(V - HCH^{\mathrm{T}}\right) V^{-1} r \tag{9}$$

$$\frac{\mathrm{d}^2\chi^2}{\mathrm{d}\alpha^2} = 2A^{\mathrm{T}} V^{-1}\left(V - HCH^{\mathrm{T}}\right) V^{-1} A. \tag{10}$$

The matrix

$$R \equiv V - HCH^{\mathrm{T}} \tag{11}$$