Available online at www.sciencedirect.com
ScienceDirect
Nuclear Instruments and Methods in Physics Research A 578 (2007) 322–328
NUCLEAR INSTRUMENTS & METHODS IN PHYSICS RESEARCH Section A
www.elsevier.com/locate/nima

Upper limit for Poisson variable incorporating systematic uncertainties by Bayesian approach

Yongsheng Zhu*

Institute of High Energy Physics, CAS, Beijing 100039, China

Received 23 March 2007; received in revised form 8 May 2007; accepted 8 May 2007
Available online 17 May 2007

Abstract

To calculate the upper limit for the Poisson observable at given confidence level with inclusion of systematic uncertainties in background expectation and signal efficiency, formulations have been established along the line of Bayesian approach. A FORTRAN program, BPULE, has been developed to implement the upper limit calculation.

PACS: 02.50.Cw; 02.50.Tt; 02.70.Pt

Keywords: Bayes statistics; Upper limit; Poisson distribution; Systematic uncertainty

1. Introduction

A group of particle physics experiments involves the search for new signal or measuring small signal at the circumstance with significant background. A limit on, or a measurement of, a physical quantity at a given confidence level is usually set by comparing a number of detected events with the expected number of background events in the "signal" region where the signal events (if exist) shall reside. How well this comparison can be made for the observed events and the expected background depends strongly on the systematic uncertainties existing in the measurement. Therefore, systematic uncertainties must be taken into consideration in the limit or confidence belt calculation.

In the frame of frequentist statistics, confidence limits are set using a Neyman construction [1]. This method suffers from so-called undercoverage and "flip-flopping" policy when the observable is close to the physics boundary, namely, the actual coverage is less than the requested coverage (confidence level) and to report a central confidence interval or an upper limit is artificially decided by the experimenter's choice. In particular, in the case when no events have been observed, this method gives no answer for the confidence interval.

Feldman and Cousins [2] proposed a new method to construct confidence interval based on likelihood ratios, which automatically provides a central confidence interval or an upper confidence limit, which is decided by the observed data itself. Therefore, it is often denoted as the "unified approach".

However, this approach also has its drawbacks. If the observable is a Poisson variable, there is a background dependence of the upper limit in the case of fewer events observed than expected background. This can lead to situation where measurements with higher background give a smaller upper limit, which is clearly undesirable. To overcome this shortcoming, Roe and Woodroofe [3] proposed a solution to this problem by using such a fact that, given an observation n, the background b cannot be large than n in any case. Therefore, the usual Poisson pdf (probability density function) should be replaced by a conditional pdf, and then this conditional pdf is used to construct the confidence interval. This approach solves the background dependence of the upper limit, however, does not satisfy all the requirements of proper coverage [4] and has problems when applied to the case of a Gaussian distribution

*Tel.: +86 10 88236053; fax: +86 10 88233083.
E-mail address: zhuys@ihep.ac.cn

0168-9002/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.nima.2007.05.116

with boundaries [5]. An extension based on a Bayesian approach with tests of coverage can be found in Ref. [6].

Along this line, a modification of the Neyman method incorporating systematic uncertainty of the signal detection efficiency has been proposed by Highland and Cousins [7], in which a "semi-Bayesian" approach is adopted, where an average over the probability of the detection efficiency is performed. This method is of limited accuracy in the limit of high relative systematic uncertainties. On the other hand, an entirely frequentist approach has been proposed for the uncertainty in the background rate prediction [8]. This approach is based on a two-dimensional confidence belt construction and likelihood ratio hypothesis testing and treats the uncertainty in the background as a statistical uncertainty rather than as a systematic one. Recently, Conrad et al. extend the method of confidence belt construction proposed in Ref. [2] to include systematic uncertainties in both the signal and background efficiencies as well as systematic uncertainty of background expectation prediction [9]. It takes into account the systematic uncertainties by assuming a pdf which parameterizes our knowledge on the uncertainties and integrating over this pdf. This method, combining classical and Bayesian elements, is referred to as semi-Bayesian approach. A FORTRAN program, POLE, has been coded to calculate the confidence intervals for a maximum of observed events of 100 and a maximum signal expectation of 50 [10].

In the frame of Bayes statistics, Narsky depicted the estimation of upper limits for Poisson statistic with the known background expectation [11,12]. Treatment of background uncertainty is discussed with the flat prior for simplified cases of background expectation distributions in Refs. [7,13].

In this paper, we use Bayesian approach to formulate the upper limit at given confidence level for the Poisson observable incorporating systematic uncertainties in the signal efficiency and background expectation. A FORTRAN program has been developed to calculate the corresponding upper limit.

2. Bayesian approach to estimate upper limit

In Bayesian approach one has to assume a prior pdf of an unknown parameter and then perform an experiment to update the prior distribution. The prior pdf reflects the experimenter's subjective degree of belief about unknown parameter before the measurement was carried out. The updated prior, called posterior pdf, is used to draw inference on unknown parameter. This updating is done with the use of Bayes theorem [14]. Assuming that n represents the number of observed events, s is the number of signal events which is unknown and to be inferred, $p(n|s)$ is the conditional pdf of observing n events with given signal s, $\pi(s)$ is the prior pdf, the Bayes theorem gives the posterior pdf:

$$h(s|n) = \frac{p(n|s)\pi(s)}{\int_0^\infty p(n|s)\pi(s)\,\mathrm{d}s}. \tag{1}$$

Here the lower limit of the integral is zero, which is the possible minimum number of signal events. Using this posterior pdf, one can calculate a Bayesian confidence interval for the signal expectation at given confidence level $\mathrm{CL} = 1 - \alpha$

$$1 - \alpha = \int_{s_\mathrm{L}}^{s_\mathrm{U}} h(s|n)\,\mathrm{d}s.$$

The upper limit of the number of signal events at given confidence level $\mathrm{CL} = 1 - \alpha$, S_{UP}, is naturally given by

$$1 - \alpha = \int_0^{S_{\mathrm{UP}}} h(s|n)\,\mathrm{d}s. \tag{2}$$

The nice feature of the Bayesian approach is that the zero value of an upper limit S_{UP} always corresponds to the zero value of confidence level $\mathrm{CL} = 1 - \alpha$, which is not necessarily true for the classical approach. The most important issue is to determine a prior pdf of the parameter. This is an issue which brings most of controversies into Bayesian methods. An important question is that if one should use an *informative* prior, i.e., a prior which incorporates results of previous experiments, or a *non-informative* prior, i.e., a prior which claims total ignorance. The major objection against informative prior is based on such argument: if we assume a prior which incorporates results of previous experiments, then our measurement will not be independent, hence, we will not be able to combine our results with previous results by taking a weighted average. Thus, we only discuss the Bayesian inference that assumes a non-informative prior for the non-negative parameter of a Poisson distribution.

For the case that in the "signal region" where the signal events resides, the number of signal events is a Poisson variable with unknown expectation s, and the number of background events is a Poisson variable with expectation b, the conditional pdf of observing n total events can be written as

$$p(n|s) = \mathrm{e}^{-(s+b)}\frac{(s+b)^n}{n!}. \tag{3}$$

To deduce the posterior pdf, one has to assume a prior pdf. Bayes stated that, the non-informative prior for any parameter must be flat [14]. This statement is not based on any strict mathematical argument, but merely his intuition. The obvious weakness of Bayes prior pdf is that if one can assume a flat distribution of an unknown parameter, then one can also assume a flat distribution for any function of this parameter; however, these two prior functions are apparently not identical. Jeffreys [15,16], Jaynes [17], and Box et al. [18] derived the non-informative prior from first principle to resolve this problem, which are proportional to $1/\theta$ and $1/\sqrt{\theta}$, respectively, where θ is the unknown parameter. Comments on these three non-informative priors can be found in Refs. [11,12]. For the pdf shown in Eq. (3), the corresponding prior pdfs are