Available online at www.sciencedirect.com
ScienceDirect
Nuclear Instruments and Methods in Physics Research A 577 (2007) 702–707
NUCLEAR INSTRUMENTS & METHODS IN PHYSICS RESEARCH Section A
www.elsevier.com/locate/nima

Optimal proton trapping strategy for a neutron lifetime experiment ☆

Kevin J. Coakley*

Statistical Engineering Division, National Institute of Standards and Technology, 325 Broadway, Boulder, C0 80305, USA

Received 27 March 2007; accepted 28 April 2007
Available online 10 May 2007

Abstract

In a neutron lifetime experiment conducted at the National Institute of Standards and Technology, protons produced by neutron decay events are confined in a proton trap. In each run of the experiment, there is a trapping stage of duration τ. After the trapping stage, protons are purged from the trap. A proton detector provides incomplete information because it goes dead after detecting the first of any purged protons. Further, there is a dead time δ between the end of the trapping stage in one run and the beginning of the next trapping stage in the next run. Based on the fraction of runs where a proton is detected, I estimate the trapping rate λ by the method of maximum likelihood. I show that the expected value of the maximum likelihood estimate is infinite. To obtain a maximum likelihood estimate with a finite expected value and a well-defined and finite variance, I restrict attention to a subsample of all realizations of the data. This subsample excludes an exceedingly rare realization that yields an infinite-valued estimate of λ. I present asymptotically valid formulas for the bias, root-mean-square prediction error, and standard deviation of the maximum likelihood estimate of λ for this subsample. Based on nominal values of λ and the dead time δ, I determine the optimal duration of the trapping stage τ by minimizing the root-mean-square prediction error of the estimate.
Published by Elsevier B.V.

PACS: 21.10.Tg; 24.80+y; 14.20.Dh; 02.50−r

Keywords: Lifetimes; Nuclear tests of fundamental interactions and symmetries; Properties of protons and neutrons; Probability theory; Stochastic processes; Statistics

1. Introduction

Ion traps play a key role in fundamental physics experiments [1]. In this paper, I focus on statistical methods for uncertainty analysis and planning of proton trap neutron lifetime experiments [1–5] and related experiments such as Ref. [6]. When a neutron decays, it produces a proton, an electron and an antineutrino. An accurate determination of the mean lifetime of the neutron is critically important for testing the fundamental theories of physics [7]. Further, the mean lifetime of the neutron is an important parameter in the astrophysical theory of big bang nucleosynthesis [8]. In a proton trap neutron lifetime experiment performed at the National Institute of Standards and Technology (NIST), a beam of neutrons passes through a detection volume. Based on measurements of the neutron flux and the proton production rate, one measures the mean lifetime of the neutron. Each run of the experiment consists of trapping stage where protons are confined in a trap [2–5], and a detection stage. The detector provides incomplete information because it goes dead after detecting the first proton. Based on the number of runs where a proton is detected, one can estimate the proton trapping rate.

In earlier work [9], this estimation problem was studied using a Bayesian method. Given a particular realization of the data (the number of runs where at least one ion (proton in this paper) is trapped), formulas for the posterior mean and posterior variance of the ion trapping rate were presented based on a prior probability model for the trapping rate. In this work, I estimate the trapping rate by the method of maximum likelihood and focus on the statistical properties of this estimate. I neglect physical

*Tel./fax: +1 303 497 3012.
E-mail address: kevin.coakley@nist.gov

0168-9002/$ - see front matter Published by Elsevier B.V.
doi:10.1016/j.nima.2007.04.172

sources of systematic error due to effects such as a time varying proton trapping rate or fluctuations in the actual trapping stage interval about the nominal value sought by the experimenter.

In Section 2, I demonstrate that the bias (expected value minus true value) and variance of the maximum likelihood estimate of the trapping rate λ are infinite. This is so because a rare realization of the data yields an infinite estimate of λ. This technical problem can be dealt with in various ways. One could quantify uncertainty by constructing confidence intervals of finite width even though the variance of the estimate is infinite. Another approach would be to introduce a stopping rule so that the experiment is continued until no protons are trapped in at least one run. I do not pursue either of these approaches here. Instead, I restrict the sample space to include only realizations of data where one observes at least one run where no protons are trapped. For realizations of data in this subsample, the maximum likelihood estimate has finite first and second moments. In Section 3, I derive asymptotically valid formulas for the bias, variance, and mean-square-error of a maximum likelihood estimate of the proton trapping rate computed from this subsample. In general, one expects estimates that are nonlinear functions of the observed data, such as the maximum likelihood estimate of the trapping rate, to be biased [10]. In Section 4, where, based on nominal values of trapping rate and dead time, I determine the trapping time that minimizes the root-mean-square prediction error of the maximum likelihood estimate of λ in the subsample of interest.

2. Statistical model

In a simulated proton trapping experiment there are many runs. During each run, I assume that the duration of the proton trapping stage τ is an adjustable constant that is known with negligible uncertainty. During the trapping stage, I assume that protons are trapped at a constant rate λ. Further, I restrict attention to the case where $\lambda > 0$. After the trapping stage, protons are purged from the trap. A proton detector provides incomplete information because it goes dead after detecting the first of any purged protons. Further, there is a fixed dead time δ between the end of the trapping stage in one run and the beginning of the next trapping stage in the next run. I assume that δ is known with negligible uncertainty. If the total time of the experiment is T, the total number of runs is

$$N_{\mathrm{run}} = \mathrm{INT}\left(\frac{T}{\tau + \delta}\right). \tag{1}$$

Above, the function $\mathrm{INT}(x)$ rounds the continuous variable x down to the nearest integer. Let n_+ be the observed number of runs where at least one proton is trapped. I model the number of protons trapped during any run as a realization of a Poisson process with expected value $\lambda\tau$. Hence, the probability that no ion is trapped for a given run is

$$p_0 = \exp(-\lambda\tau). \tag{2}$$

The maximum likelihood estimate of p_0 is

$$\hat{p}_0 = 1 - \frac{n_+}{N_{\mathrm{run}}} \tag{3}$$

where n_+ is the number of runs where at least one proton is trapped. Thus, the maximum likelihood estimate of λ is

$$\hat{\lambda} = -\frac{1}{\tau}\ln \hat{p}_0 = -\frac{1}{\tau}\ln\left(1 - \frac{n_+}{N_{\mathrm{run}}}\right). \tag{4}$$

Since n_+ is a binomial random variable, the probability that $n_+ = k$ is $P(k)$, where

$$P(k) = \frac{N_{\mathrm{run}}!}{(N_{\mathrm{run}} - k)!k!}(1 - p_0)^k p_0^{N_{\mathrm{run}}-k}. \tag{5}$$

Hence, the expected value of the maximum likelihood estimate of λ is

$$E(\hat{\lambda}) = -\frac{1}{\tau}\sum_{k=0}^{N_{\mathrm{run}}} P(k)\ln\left(1 - \frac{k}{N_{\mathrm{run}}}\right). \tag{6}$$

Similarly, the expected squared value of the estimate is

$$E(\hat{\lambda}^2) = \frac{1}{\tau^2}\sum_{k=0}^{N_{\mathrm{run}}} P(k)\left(\ln\left(1 - \frac{k}{N_{\mathrm{run}}}\right)\right)^2. \tag{7}$$

For $\lambda > 0$, $P(N_{\mathrm{run}}) = (1 - p_0)^{N_{\mathrm{run}}} > 0$, and both the expected value (first moment) and expected squared value (second moment) of $\hat{\lambda}$ are infinite.The variance of $\hat{\lambda}$, $VAR(\hat{\lambda})$, is not defined because

$$VAR(\hat{\lambda}) = E(\hat{\lambda}^2) - (E(\hat{\lambda}))^2 \tag{8}$$

and both terms on the right-hand side of Eq. (8) are infinite.

To ensure that both $E(\hat{\lambda})$ and $E(\hat{\lambda}^2)$ are finite, I restrict the sample space to realizations of the data where $n_+ < N_{\mathrm{run}}$. From a practical point of view, this means that realizations of data where $n_+ = N_{\mathrm{run}}$ would be ignored. For neutron lifetime experiments of current interest, the probability that $n_+ = N_{\mathrm{run}}$ is negligible provided that τ is judiciously chosen. Hence, this subsampling restriction does not significantly affect data collection procedures for neutron lifetime experiments of current interest. In this subsample, the discrete probability density function for allowed realizations of $n_+ = 0, 1, \ldots, N_{\mathrm{run}} - 1$ is $P_*(k)$, where

$$P_*(k) = \frac{P(k)}{1 - P(N_{\mathrm{run}})}. \tag{9}$$

For this subsample, the first two moments of the maximum likelihood estimate are

$$E(\hat{\lambda}) = -\frac{1}{\tau}\sum_{k=0}^{N_{\mathrm{run}}-1} P_*(k)\ln\left(1 - \frac{k}{N_{\mathrm{run}}}\right) \tag{10}$$