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The statistics of multi-photoelectron pulse-height distributions

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Abstract

Multi-photoelectron pulse-height distributions feature in the scientific literature on light detectors, and yet there appears to have been no attempt to describe the statistics of such spectra. Formulas for pdfs are derived, assuming Poisson statistics, but more importantly for single-electron response curves of any description—analytical or experimental. The pdf generated by a pulsed light source, detected by a PMT of high d_1 gain, agrees with statistical predictions.

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1. Introduction

Increasingly in the literature, and particularly in this journal, we are presented with multi-photoelectron pulse-height distributions (MPHD) in which peaks due to 1, 2, … N photoelectrons are clearly seen. In the proceedings of the last three Beaune conferences on photodetection [1], no less than 24 of the presentations include such spectra. It is generally understood that the resolution of this class of detector is somehow linked to N, the larger the N, the better is the resolution. The first MPHD to appear in the literature is probably the one due to Morton et al. [2] at RCA. Their newly developed vacuum photomultiplier (PMT) with a GaP, high d_1 gain, first stage showed the imprint of five photoelectrons. Since that time, further detectors capable of even higher resolution have been developed. These detectors include: microchannel plate photomultipliers (MCPMT), hybrid photomultipliers (HPD), and more recently, several different types of silicon PMTs (SiPM) based on multi-pixel avalanche photodiodes. The last two mentioned are capable of resolving up to 20 peaks. Detailed descriptions of all these devices and their performance capabilities can be found in Ref. [1].

A well-resolved single-electron response (SER) is a prerequisite for a highly structured MPHD. In practical terms, an SER with a peak-to-valley ratio in excess of three and a resolution of $<60\%$ is sufficient for this purpose. Despite the obvious importance of the MPHD to a wide range of detectors, there is, to the author's knowledge, no published statistical treatment of the subject that is wholly satisfactory. In one paper [3] concerning PMTs, the theoretical treatment begins with the statement 'the charge amplification process initiated by one photoelectron can be approximated by a Gaussian distribution.' This is certainly not the case for conventional PMTs where the SERs are always asymmetrical. The method first used by Ranucci et al. [4] and subsequently by Ankowski et al. [5] is more satisfactory because the simulation assumes a combination of an exponential and a Gaussian. There is, however, no physics underlying any of these assumptions. The existing procedures are closer to curve fitting than they are to statistical predictions—which is the purpose of the present study.

The statistical treatment that has been adopted for the multiplication (gain) process is general. It allows for the use of any mathematical distribution, or, more importantly, even an experimentally measured SER.

For completeness, although it is of limited utility, we consider as a special case, the assumption of Poisson

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statistics for the multiplier. Lombard and Martin [6] showed for an ideal multiplier, all of whose stages obey Poisson statistics, that the first stage of multiplication is the most important where resolution is concerned. If the first stage gain is sufficiently high, say greater than five, then the subsequent stages will have negligible effect on the dispersion of the anode signal. Since, in this study, we are dealing with devices with near noiseless gain, we can justifiably use a model with just a single, very high gain, stage of multiplication for this special case.

It is shown how any statistical MPHD can be deconvoluted into its N-fold components. This process provides an insight into the underlying structure of the MPHD.

2. Statistical considerations

Light sources relevant to particle and astrophysics are predominantly Poisson in their statistical behaviour and the photoelectric effect is certainly binomial. It is readily shown that the combined process is Poisson, so there is no loss in generality by assuming Poisson statistics for photoelectron emission.

The approach used in this work is similar to that followed by Ref. [6], where the use of generating functions is central to arriving at the final results. We cascade two distributions A and B. A describes the light detection statistics, assumed to be Poisson, while B refers to the statistics of the multiplier, which for the moment is unspecified. A happens first and B operates on the outcome of A. We derive an expression for the combined statistics for any distribution representing B. Given the SER of a photon detecting device, whether it is analytical or experimentally determined, we predict the MPHD for all the detector types.

The probability distributions for A and B are represented by $p(n)$ and $q(s)$, respectively. The generating function, $G_B(u)$, is by definition

$$G_B(u) = q(0) + q(1)u + q(2)u^2 + \cdots + q(s)u^s \tag{1}$$

where a finite number of terms, $s+1$, has been assumed to make the point that Eq. (1) can equally represent a set of experimental points, $q(s)$, or an infinite set of probabilities, as for a Poisson distribution for example. The generating function for the cascaded process [7] is, in any case

$$G_{AB}(u) = G_A[G_B(u)] \tag{2}$$

$$G_{AB}(u) = p(0) + p(1)[G_B(u)] + p(2)[G_B(u)]^2 + p(3)[G_B(u)]^3 + \cdots \tag{3}$$

$G_{AB}(u)$ may also be written in terms of the cascaded probability distribution, $P(r)$, which is the distribution required:

$$G_{AB}(u) = P(0) + P(1)u + P(2)u^2 + P(3)u^3 + \cdots \tag{4}$$

Differentiation of the generating functions is denoted by a superscript in parentheses, for example $G_{AB}^{(r)}(u)$ is $G_{AB}(u)$ differentiated r times, but a superscript without brackets indicates an index. By repeated differentiation of Eq. (4) we have

$$G_{AB}^{(r)}(0) = r!P(r). \tag{5}$$

For presentation purposes, we define $G_{AB}^{(0)}(u)$ to be the undifferentiated function $G_{AB}(u)$. We now replace the $p(n)$ terms in Eq. (3) by Poisson probabilities, for a mean of m_1 photoelectrons, arriving at the combined generating function, $G_{AB}(u)$; $q(s)$ is embodied in $G_B(u)$ but still undefined.

$$G_{AB}(u) = e^{-m1} + e^{-m1} m_1 G_B(u) + e^{-m1}\frac{\{m_1 G_B(u)\}^2}{2!} + e^{-m1}\frac{\{m_1 G_B(u)\}^3}{3!} + \cdots$$

$$G_{AB}^{(0)}(u) = e^{-m1}\exp\{m_1 G_B(u)\} \tag{6}$$

$$G_{AB}^{(1)}(u) = m_1 G_B^{(1)}(u) G_{AB}^{(0)}(u)$$
$$G_{AB}^{(2)}(u) = m_1\{G_B^{(2)}(u)\, G_{AB}^{(0)}(u) + G_B^{(1)}(u)\, G_{AB}^{(1)}(u)\}$$
$$G_{AB}^{(3)}(u) = m_1\{G_B^{(3)}(u)\, G_{AB}^{(0)}(u) + 2G_B^{(2)}(u)\, G_{AB}^{(1)}(u) + G_B^{(1)}(u)\, G_{AB}^{(2)}(u)\}$$

$$G_{AB}^{(r+1)}(u) = m_1\left\{\sum_{k=0}^{r}\frac{r!}{k!(r-k)!}G_B^{(r+1-k)}(u)\, G_{AB}^{(k)}(u)\right\} \tag{7}$$

From Eq. (5), $G_{AB}^{(r+1)}(0) = (r+1)!P(r+1)$, $G_B^{(r+1-k)}(0) = (r+1-k)!q(r+1-k)$ and $G_{AB}^{(k)}(0) = k!P(k)$ leading to

$$P(r+1) = \frac{m_1}{r+1} - \sum_{k=0}^{r}(r+1-k)\, q(r+1-k)P(k). \tag{8}$$

Eq. (8) is subject to $(r+1-k)\leqslant s$ from Eq. (1). From Eqs. (4) and (6), $P(0) = G_{AB}^{(0)}(0) = \exp\{-m_1(1-q(0))\}$. Knowing $P(0)$, we can calculate $P(1)$ from Eq. (8), which gives $P(1) = m_1 q(1)\exp\{-m_1(1-q(0))\}$. Continuing this iterative process will yield the complete set of $P(r)$.

Consider the special case for which Poisson statistics apply to the first stage of the multiplier, that is $q(r) = e^{-m2}m_2\, r/r!$. Substituting into Eq. (8) we have

$$P(r+1) = \frac{m_1 m_2\, e^{-m2}}{r+1}\sum_{k=0}^{r} m_2^{r-k}\frac{P(k)}{(r-k)!}. \tag{9}$$

3. Folded distributions

Eqs. (8) and (9) provide the envelope of MPHD. The analysis that follows will allow us to probe this envelope to reveal its structure in terms of: one-photoelectron, two-photoelectron … N-photoelectron pulse-height distributions, all suitably weighted. Although these distributions are contrived, in that they do not exist in isolation, they are useful devices for revealing the structure of the MPHD. Given any probability distribution, $q(s)$, with s finite, we generate the pdf for the distribution obtained by folding $q(s)$ with itself N times, where $Q_N(r)$, the N-folded pdf, is a function of $q(s)$ only. We define $Q_1(s) = q(s)$ to represent the prime distribution (the SER); $Q_2(2s)$ is the distribution obtained by folding $q(s)$ with itself and $Q_3(3s)$ results from