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Investigating the domain of validity of the Gubser solution to the Boltzmann equation

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Abstract

We study the evolution of the one particle distribution function that solves exactly the relativistic Boltzmann equation within the relaxation time approximation for a conformal system undergoing simultaneously azimuthally symmetric transverse and boost-invariant longitudinal expansion. We show, for arbitrary values of the shear viscosity to entropy density ratio, that the distribution function can become negative in certain kinematic regions of the available phase space depending on the boundary conditions. For thermal equilibrium initial conditions, we determine numerically the physical boundary in phase space where the distribution function is always positive definite. The requirement of positivity of this particular exact solution restricts its domain of validity, and it imposes physical constraints on its applicability.

Keywords: Boltzmann equation; Statistical mechanics; Relativistic kinetic theory; Relativistic transport

1. Introduction

The dynamics and transport properties of rarefied gases and fluids are usually described in terms of the Boltzmann equation. The Boltzmann equation is a partial integro-differential equation for the distribution function $f(x, p)$. In general this equation is solved numerically, and very few exact solutions are known in the literature. Nevertheless, for highly symmetric systems it is possible to solve this equation analytically under certain approximations for the collisional

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kernel [1]. Using a relaxation time approximation (RTA) for the collisional kernel, the relativistic Boltzmann equation has been solved exactly for systems undergoing Bjorken flow [2] and Gubser flow [3,4]. The exact solution for the Bjorken flow has been useful to understand aspects of the isotropization and thermalization problem of a plasma formed by quarks and gluons (QGP) [2–10]. In contrast to Bjorken flow [11], where the system expands in boost invariant fashion only along one direction (the "longitudinal" direction), Gubser flow [12,13] describes systems that undergo additionally simultaneous azimuthally symmetric expansion in the transverse directions. The solution of the Boltzmann equation for the Gubser flow [3,4] was found by exploiting the $SO(3) \otimes SO(1,1) \otimes Z_2$ symmetry of the flow velocity profile [12,13] which becomes manifest when mapping Minkowski space, $R^3 \otimes R$, conformally onto de Sitter space times a line, $dS_3 \otimes R$ [12,13]. In Refs. [3,4] it was noticed that the resulting solutions for moments of the distribution function, such as the energy density or temperature, can become complex and therefore physically meaningless when propagating backwards in the de Sitter time (see discussion in Appendix B of Ref. [4]). In this work we revisit this issue and find that the unphysical behavior of the moments of the distribution function found in Ref. [4] is rooted in a violation of the positivity of the distribution function in some regions of phase space when propagating the solution of the Boltzmann equation for equilibrium initial conditions backward in the de Sitter time. In Minkowski coordinates, this translates to the distribution function becoming negative in certain momentum ranges at the outer edge of the spatial density profile at fixed time.

This work is organized as follows: in Section 2 we review the procedure used in Refs. [3,4] to find the exact solution of the Boltzmann equation [3,4]. In Section 3 we show numerical results for the phase space evolution of the distribution function. Our conclusions are presented in Section 4.

2. The analytical solution of the RTA Boltzmann equation for the Gubser expansion

In this section we briefly review the derivation of the exact solution to the RTA Boltzmann equation that is invariant under the group of symmetries of the Gubser flow, i.e., under the $SO(3)_q \otimes SO(1,1) \otimes Z_2$ group ("Gubser group") [12,13]. For additional technical details of the method discussed here we refer the reader to Ref. [4].

We use the following notation. The metric signature is taken to be the "mostly plus" convention. In Minkowski space the line element is written in Milne coordinates $x^\mu = (\tau, r, \phi, \varsigma)$ as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -d\tau^2 + dr^2 + r^2 d\phi^2 + \tau^2 d\varsigma^2, \tag{1}$$

where the longitudinal proper time τ, the spacetime rapidity ς, the transverse radius r and the azimuthal angle ϕ are defined in terms of the usual cartesian coordinates (t, x, y, z) as

$$\begin{aligned} \tau &= \sqrt{t^2 - z^2}, \qquad \varsigma = \tanh^{-1}\left(\frac{z}{t}\right) \\ r &= \sqrt{x^2 + y^2}, \qquad \phi = \arctan\left(\frac{y}{x}\right). \end{aligned} \tag{2}$$

The flow velocity u^μ is normalized as $u_\mu u^\mu = -1$.

2.1. The Gubser flow

The dynamics of an expanding conformal fluid in Minkowski space can be understood in terms of a static conformal fluid defined in a particular curved space. In Minkowski space, the