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Solving the sign problems of the massless lattice Schwinger model with a dual formulation

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Abstract

We derive an exact representation of the massless Schwinger model on the lattice in terms of dual variables which are configurations of loops, dimers and plaquette occupation numbers. When expressed with the dual variables the partition sum has only real and positive terms also when a chemical potential or a topological term are added – situations where the conventional representation has a complex action problem. The dual representation allows for Monte Carlo simulations without restrictions on the values of the chemical potential or the vacuum angle.

1. Introduction

Since its initial formulation three decades ago lattice QCD has developed into a reliable quantitative tool for studying many low energy phenomena in QCD. However, one important issue where the lattice approach has essentially failed so far is its application to QCD at finite density. The reason is that at finite chemical potential μ the fermion determinant becomes complex and cannot be used as a probability weight in a Monte Carlo simulation. This is known as the “complex action problem” or “sign problem”. The complex action problem is not specific for QCD or for theories with fermions, but is a generic feature of many quantum field theories at finite

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density both in the lattice and the continuum formulation. Furthermore, not only finite chemical potential gives rise to a complex action problem, but also the addition of a topological term.

For overcoming the complex action problem in lattice simulations several approaches have been discussed, such as reweighting, various series expansions, the reformulation with alternative variables and simulations based on stochastic differential equations. Reviews at the yearly lattice conferences [1] give an overview about the progress in recent years.

Probably the most elegant approach to the complex action problem is to rewrite a lattice field theory in terms of new degrees of freedom, often referred to as "dual variables", such that in the new representation the partition sum has only real and positive contributions. The dual variables usually turn out to be loops for matter fields and surfaces for gauge fields. The key problem of this approach is that there is no general recipe for finding a dual representation of a given theory. In particular non-abelian gauge fields are difficult and so far no convincing approach has emerged. The situation is different for theories with abelian gauge fields where for the case of their interaction with bosonic matter several interesting results based on dualization [2,3] were presented in recent years.

Coupling fermions to the gauge fields brings in additional difficulties with extra signs for the matter loops coming from the Grassmann nature of the fermions and from traces over gamma-matrices. In 2 dimensions, however, the understanding of the traces over gamma-matrices as they appear in dual fermion loops is much more advanced [4], such that the Schwinger model, i.e., QED in 2 dimensions, seems to be an interesting candidate for further developing the dual approach for theories of gauge fields interacting with fermions. Indeed, some results for at least partial dualizations of the lattice Schwinger model were discussed in the literature [5–7], but no convincing complete solution of the sign problem was presented so far.

In this article we show that for the lattice Schwinger model with massless staggered fermions the complex action problem can be solved completely by exactly rewriting the model to a dual representation. We consider both types of complex action problems, coming from finite density and the addition of a topological term. The dual variables are closed oriented loops for the fermions, with the gauge fields being represented by integer valued plaquette occupation numbers. The constraints for these dual degrees of freedom again allow for the interpretation of the dual gauge field variables as surfaces in 2 dimensions. We show that in the dual form the partition sum has only real and positive contributions. Furthermore we discuss the dual formulation of various observables and briefly address possible update strategies.

We stress that the dual representation discussed here not only provides an interesting step towards solving complex action problems for theories of gauge fields interacting with fermions, but that it will also be useful for addressing some of the open questions concerning the phase structure and universality class of the massless lattice Schwinger model with staggered fermions [8,9].

2. Conventional form of the one flavor model with topological term

We begin with the case of the massless one-flavor model with a topological term. In the conventional representation the corresponding partition sum is given by

$$Z = \int \mathcal{D}[U]\mathcal{D}\big[\,\overline{\psi},\psi\,\big]\, e^{-\,S_G[U]\,-i\,\theta\,Q[U]\,-\,S_\psi[U,\overline{\psi},\psi]}\,. \tag{1}$$

The dynamical degrees of freedom are U(1)-valued link variables $U_\nu(n) = \exp(i A_\nu(n)),\ A_\nu(n) \in [-\pi, \pi]$. Here $n = (n_1, n_2)$ denotes the sites of a 2-dimensional $N_S \times N_T$ lattice and $\nu = 1, 2$