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Graßmannian integrals as matrix models for non-compact Yangian invariants

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Abstract

In the past years, there have been tremendous advances in the field of planar $\mathcal{N} = 4$ super Yang–Mills scattering amplitudes. At tree-level they were formulated as Graßmannian integrals and were shown to be invariant under the Yangian of the superconformal algebra $\mathfrak{psu}(2,2|4)$. Recently, Yangian invariant deformations of these integrals were introduced as a step towards regulated loop-amplitudes. However, in most cases it is still unclear how to evaluate these deformed integrals. In this work, we propose that changing variables to oscillator representations of $\mathfrak{psu}(2,2|4)$ turns the deformed Graßmannian integrals into certain matrix models. We exemplify our proposal by formulating Yangian invariants with oscillator representations of the non-compact algebra $\mathfrak{u}(p,q)$ as Graßmannian integrals. These generalize the Brezin–Gross–Witten and Leutwyler–Smilga matrix models. This approach might make elaborate matrix model technology available for the evaluation of Graßmannian integrals. Our invariants also include a matrix model formulation of the $\mathfrak{u}(p,q)$ R-matrix, which generates non-compact integrable spin chains.

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1. Introduction

The maximally supersymmetric Yang–Mills theory in four-dimensions, for short $\mathcal{N}=4$ SYM, is a remarkably rich mathematical model. Even more so in the planar limit where the theory is conjectured to be integrable. By now this integrability is well established for the spectral problem of anomalous dimensions, see the comprehensive review series [1]. Less is known about integrability for scattering amplitudes. However, at tree-level the amplitudes can be encoded as surprisingly simple formulas, so-called Graßmannian integrals [2,3], see also [4]. The mere existence of such formulas already hints at an underlying integrable structure. Furthermore, it was shown that tree-level amplitudes are invariant under the Yangian of the superconformal algebra $\mathfrak{psu}(2,2|4)$ [5]. For the Graßmannian integral formulation this was achieved in [6,7]. The appearance of this infinite-dimensional Yangian algebra is synonymous with integrability. Later, it was observed that the tree-level amplitudes allow for multi-parameter deformations while maintaining Yangian invariance. These deformations are of considerable interest as they relate the four-dimensional scattering problem to the two-dimensional quantum inverse scattering method. Furthermore, they might regulate infrared divergences at loop-level [8,9].

As in the undeformed case, the deformed tree-level amplitudes can be nicely packaged as Graßmannian integrals [10,11]. Let us briefly review this formulation. The *Graßmannian* $\mathrm{Gr}(N,K)$ is the space of all K-dimensional linear subspaces of $\mathbb{C}^N$. The entries of a $K\times N$ matrix C provide "homogeneous" coordinates on this space. The transformation $C\mapsto VC$ with $V\in GL(K)$ corresponds to a change of basis within a given subspace, and thus it does not change the point in the Graßmannian. This allows us to describe a generic point in $\mathrm{Gr}(N,K)$ by the "gauge fixed" matrix

$$C=\left(1_{K\times K}\,\middle|\,\mathcal{C}\right)\quad\text{with}\quad \mathcal{C}=\begin{pmatrix}C_{1K+1} & \cdots & C_{1N}\\ \vdots & & \vdots\\ C_{KK+1} & \cdots & C_{KN}\end{pmatrix}. \tag{1.1}$$

The amplitudes are labeled by the number of particles N and the degree of helicity violation K. Amplitudes with $K=2$ are maximally helicity violating (MHV). The deformed N-point N^{K-2}MHV tree-level amplitude is given by the *Graßmannian integral*

$$\mathcal{A}_{N,K}=\int \mathrm{d}\mathcal{C}\,\frac{\delta^{4K|4K}(C\mathcal{W})}{(1,\ldots,K)^{1+v_K^+-v_1^-}\cdots(N,\ldots,K-1)^{1+v_{K-1}^+-v_N^-}} \tag{1.2}$$

with the holomorphic $K(N-K)$-form $\mathrm{d}\mathcal{C}=\bigwedge_{k,l}\mathrm{d}C_{kl}$. In this formula $(i,\ldots,i+K-1)$ denotes the minor of the matrix C consisting of the consecutive columns $i,\ldots,i+K-1$. These are counted modulo N such that they are in the range $1,\ldots,N$. The kinematics of the j-th particle is encoded in a supertwistor with components $\mathcal{W}_A^j$, where A is a fundamental $\mathfrak{gl}(4|4)$ index. The $2N$ deformation parameters $\{v_i^+,v_i^-\}$ have to obey the constraints

$$v_{i+K}^+=v_i^- \tag{1.3}$$

for $i=1,\ldots,N$. Then the Graßmannian integral (1.2) is invariant under the Yangian of $\mathfrak{psu}(2,2|4)$, where the generators of the algebra act on the supertwistors. In the undeformed case $v_i^\pm=0$, the proper integration contour for (1.2) is known and the integral can be evaluated by means of a multi-dimensional residue theorem [2,4]. In the deformed case, the evaluation is much more involved due to branch cuts of the integrand. Most notably there are partial results on the 6-point NMHV amplitude [10]. However, finding an appropriate multi-dimensional integration contour for the evaluation of (1.2) is still a pressing open problem.