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Exploring arbitrarily high orders of optimized perturbation theory in QCD with $n_f \to 16\frac{1}{2}$

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Abstract

Perturbative QCD with n_f flavours of massless quarks becomes simple in the hypothetical limit $n_f \to 16\frac{1}{2}$, where the leading β-function coefficient vanishes. The Banks–Zaks (BZ) expansion in $a_0 \equiv \frac{8}{321}(16\frac{1}{2} - n_f)$ is straightforward to obtain from perturbative results in $\overline{\text{MS}}$ or any renormalization scheme (RS) whose n_f dependence is 'regular'. However, 'irregular' RS's are perfectly permissible and should ultimately lead to the same BZ results. We show here that the 'optimal' RS determined by the Principle of Minimal Sensitivity does yield the same BZ-expansion results when all orders of perturbation theory are taken into account. The BZ limit provides an arena for exploring optimized perturbation theory at arbitrarily high orders. These explorations are facilitated by a 'master equation' expressing the optimization conditions in the fixed-point limit. We find an intriguing strong/weak coupling duality $a \to {a^*}^2/a$ about the fixed point a^*.

1. Introduction

The initial impulse for these investigations was a concern with the compatibility of the Banks–Zaks (BZ) expansion [1–4] with renormalization-scheme (RS) invariance [5]. In dimensional regularization the β function naturally has a term $-\epsilon a$ which strongly affects any zero near the origin. Can one safely take $\epsilon \to 0$ first and *then* take $n_f \to 16\frac{1}{2}$, or do these limits somehow clash? Our results here basically resolve those concerns; the BZ expansion appears to be

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fully compatible with RS invariance in the sense that "optimized perturbation theory" (OPT) [6], which enforces local RS invariance in each order, ultimately yields the same BZ results.

The BZ expansion is normally discussed only within a restricted class of 'regular' schemes. However, infinitely many schemes – and in some sense most schemes – are not 'regular.' In particular, the "optimal" scheme is not. In 'regular' schemes one needs only k terms of the perturbation series to obtain k terms of the BZ expansion, but in other schemes the information needed is distributed among higher-order terms [7]. In general all orders are required. Turning that observation around, the BZ expansion can be viewed as a "playground" in which one can analytically investigate arbitrarily high orders of OPT in QCD. Admittedly, this adopts the "drunk-under-the-lamppost" principle of looking, not where we really want to, but where there is enough light to make a search. The deep and difficult issues that we would like to study – "renormalons" and factorially growing coefficients – are simply absent in the BZ limit. Nevertheless, we believe our search provides some interesting insights and employs some methods that may have wider applicability.

Infrared fixed points and divergent perturbation series were no part of the motivation for OPT [6], but OPT has important consequences for both these topics. Fixed points in OPT are discussed in Refs. [8–14]. Such infrared behaviour was found for $R_{e^+e^-}$ at third order for all n_f [9,10], though error estimates at low n_f are large.[1]

The role of OPT in taming high-order perturbation theory was investigated in Ref. [15]. A toy example, involving an alternating factorial series, showed that even when the perturbation series is badly divergent in any fixed RS, the sequence of optimized approximants can converge. This "induced convergence" mechanism (related to the idea of "order-dependent mappings" [16]) has been shown to operate [17] in the anharmonic oscillator and ϕ^4 field theories in the variational perturbation theory of Refs. [18–20]. In QCD "induced convergence" of OPT has been investigated in the large-b approximation [21]. It has also been shown [22] that adjusting the renormalization scale with increasing order — which happens naturally in OPT [15] — can indeed have dramatic and beneficial effects on series behaviour. In the present paper we work in the small-b approximation (the BZ limit), where the issues are rather different. In particular, the role of optimizing other aspects of the RS, besides the renormalization scale, come to the fore.

The plan of the paper is as follows. Following some preliminaries in Sect. 2, the BZ expansion, as obtained from 'regular' schemes, is summarized in Sect. 3, and we note that it is suffices to consider two infrared quantities, $\mathcal{R}^*$ and γ^*. Sect. 4 briefly reviews OPT. Sect. 5 presents OPT results in the BZ limit, up to 19th order. Sect. 6 describes analytic methods for studying OPT at arbitrarily high orders. It also introduces a crude approximation, "NLS," and a better approximation, "PWMR." These approximations, applied to the BZ limit, are explored in detail in Sects. 7 and 8. From these results we see that OPT, taken to all orders, does reproduce the expected BZ-limit results, and we gain some insight into how OPT's subtle features conspire to produce accurate results and a rather well-behaved series for $\mathcal{R}^*$. In Sect. 9 we show that all-orders OPT reproduces higher terms in the BZ expansion correctly, and in Sect. 10 we point out an intriguing $a \to a^{*2}/a$ duality. Our conclusions are summarized in Sect. 11. (Two appendices discuss (a) some subtleties associated with the critical exponent γ^* [23–25] and (b) the pinch mechanism [14], which is a way that a finite infrared limit can occur in OPT without a fixed point. This mechanism is probably not directly relevant in the BZ limit, though it nearly is.)

[1] Also, other physical quantities behave rather differently [11]. The idea [4,7] that the BZ expansion can be extrapolated, crudely, to low n_f no longer seems tenable [14]. The "freezing" behaviour at small n_f, confirmed at fourth order [12–14], seems instead to stem from somewhat different physics.