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Dual representation for the generating functional of the Feynman path-integral

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Abstract

The generating functional for scalar theories admits a representation which is dual with respect to the one introduced by Schwinger, interchanging the role of the free and interacting terms. It maps $\int V(\delta_J)$ and $J\Delta J$ to $\delta_{\phi_c}\Delta\delta_{\phi_c}$ and $\int V(\phi_c)$, respectively, with $\phi_c = \int J\Delta$ and Δ the Feynman propagator. Comparing the Schwinger representation with its dual version one gets a little known relation that we prove to be a particular case of a more general operatorial relation. We then derive a new representation of the generating functional $T[\phi_c] = W[J]$ expressed in terms of covariant derivatives acting on 1

$$T[\phi_c] = \frac{N}{N_0}\exp(-U_0[\phi_c])\exp\Big(-\int V(\mathcal{D}^-_{\phi_c})\Big)\cdot 1$$

where $\mathcal{D}^{\pm}_{\phi}(x) = \mp\Delta\frac{\delta}{\delta\phi}(x) + \phi(x)$. The dual representation, which is deeply related to the Hermite polynomials, is the key to express the generating functional associated to a sum of potentials in terms of factorized generating functionals. This is applied to renormalization, leading to a factorization of the counterterms of the interaction. We investigate the structure of the functional generator for normal ordered potentials and derive an infinite set of relations in the case of the potential $\frac{\lambda}{n!}:\phi^n:$. Such relations are explicitly derived by using the Faà di Bruno formula. This also yields the explicit expression of the generating functional of connected Green's functions.

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1. Introduction and summary

The path-integral is a basic tool in several research fields, quantum mechanics, quantum field theory, statistical mechanics etc. The original idea is due to Dirac who observed a deep analogy between the Hamilton–Jacobi theory and the quantum transition amplitudes, proposing the relation [1]

$$\langle q,t|Q,T\rangle \sim e^{-\frac{i}{\hbar}\int_T^t dtL} \ . \tag{1.1}$$

Subsequently, Feynman developed Dirac's idea, starting from an infinitesimal version of (1.1), another key step toward the path-integral formulation [2].

A relevant progress in the path-integral approach is due to Schwinger who expressed the generating functional in the form

$$W[J]=\frac{N}{N_0}\exp\Big(-\int V\Big(\frac{\delta}{\delta J}\Big)\Big)\exp(-Z_0[J]) \ , \tag{1.2}$$

where V is the potential, J the external source and $\exp(-Z_0[J])$ the generating functional of the free theory. It turns out that the Schwinger representation admits the dual representation

$$W[J]=\frac{N}{N_0}\exp(-Z_0[J])\exp\Big(\frac{1}{2}\frac{\delta}{\delta J}\Delta^{-1}\frac{\delta}{\delta J}\Big)\exp\Big[-\int d^Dx V\Big(\int d^Dz J(z)\Delta(z-x)\Big)\Big] \ , \tag{1.3}$$

with $\Delta(y-x)$ the Feynman propagator. Such a dual representation leads to consider the field

$$\phi_c(x)=\int d^Dy J(y)\Delta(y-x) \ , \tag{1.4}$$

rather than J and then defining $T[\phi_c]=W[J]$.

As we will see, this leads to represent the path-integral operator as an operator acting by functional derivatives. Namely, for any functional $F[\phi]$, it holds

$$N_0\int D\phi \exp\Big(-\frac{1}{2}\phi\Delta\phi\Big)F[\phi]=\exp\Big(\frac{1}{2}\frac{\delta}{\delta\chi}\Delta\frac{\delta}{\delta\chi}\Big)F[\chi]|_{\chi=0} \ . \tag{1.5}$$

This is a consequence of the general relation

$$\langle 0|T F[\hat{\phi}+g]|0\rangle=\exp\Big(\frac{1}{2}\frac{\delta}{\delta g}\Delta\frac{\delta}{\delta g}\Big)F[g] \ , \tag{1.6}$$

derived in Sec. 2. We then will derive a new representation of the generating functional that simplifies considerably the computations. Namely, in Sec. 6, we will see that $T[\phi_c]$ can be expressed in terms of covariant derivatives acting on 1, that is

$$T[\phi_c]=\frac{N}{N_0}\exp(-U_0[\phi_c])\exp\Big(-\int V(\mathcal{D}^-_{\phi_c})\Big)\cdot 1 \ , \tag{1.7}$$

where

$$\mathcal{D}^{\pm}_{\phi}(x)=\mp\Delta\frac{\delta}{\delta\phi}(x)+\phi(x) \ . \tag{1.8}$$

We will derive the little-known representation (1.3), reported in Fried's book [3], in two different ways. In Sec. 2 Eq. (1.3) is derived using the path-integral and the operator formalism. Then, in