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Nuclear Physics B 910 (2016) 240–258

NUCLEAR PHYSICS B

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Weak associativity and deformation quantization

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Received 19 June 2016; accepted 3 July 2016

Editor: Hubert Saleur

Abstract

Non-commutativity and non-associativity are quite natural in string theory. For open strings it appears due to the presence of non-vanishing background two-form in the world volume of Dirichlet brane, while in closed string theory the flux compactifications with non-vanishing three-form also lead to non-geometric backgrounds. In this paper, working in the framework of deformation quantization, we study the violation of associativity imposing the condition that the associator of three elements should vanish whenever each two of them are equal. The corresponding star products are called alternative and satisfy important for physical applications properties like the Moufang identities, alternative identities, Artin's theorem, etc. The condition of alternativity is invariant under the gauge transformations, just like it happens in the associative case. The price to pay is the restriction on the non-associative algebra which can be represented by the alternative star product, it should satisfy the Malcev identity. The example of nontrivial Malcev algebra is the algebra of imaginary octonions. For this case we construct an explicit expression of the non-associative and alternative star product. We also discuss the quantization of Malcev–Poisson algebras of general form, study its properties and provide the lower order expression for the alternative star product. To conclude we define the integration on the algebra of the alternative star products and show that the integrated associator vanishes.

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http://dx.doi.org/10.1016/j.nuclphysb.2016.07.004

1. Introduction

In the canonical formulation of quantum mechanics the physical observables are represented by the hermitian linear operators $\hat{f}$ acting on the Hilbert space. The composition of quantum mechanical operators in general is non-commutative, $\hat{f}\hat{g} \neq \hat{g}\hat{f}$, yielding the uncertainty principle, but should be necessarily associative, $\hat{f}(\hat{g}\hat{h}) = (\hat{f}\hat{g})\hat{h}$. The later property implies two important identities involving the commutator of operators, the Leibniz rule, $[\hat{f}\hat{g}, \hat{h}] = [\hat{f}, \hat{h}]\hat{g} + \hat{f}[\hat{g}, \hat{h}]$, and the Jacobi Identity,

$$[\hat{f}, [\hat{g}, \hat{h}]] + [\hat{h}, [\hat{f}, \hat{g}]] + [\hat{g}, [\hat{h}, \hat{f}]] \equiv 0. \tag{1}$$

The time evolution in the Heisenberg picture is postulated by the operator equation

$$i\hbar \frac{d}{dt}\hat{f} = i\hbar \frac{\partial \hat{f}}{\partial t} + [\hat{H}, \hat{f}], \tag{2}$$

where $\hat{H}$ stands for the hamiltonian operator. The Leibniz rule and the Jacobi identity guaranty the consistency of the quantum dynamics, meaning that the time evolution will preserve the algebra of physical observables. In particular, if $\hat{f}$ and $\hat{g}$ are two integrals of motion, $[\hat{H}, \hat{f}] = [\hat{H}, \hat{g}] = 0$, the commutator $[\hat{f}, \hat{g}]$ is also an integral of motion,

$$i\hbar \frac{d}{dt}[\hat{f}, \hat{g}] = [\hat{H}, [\hat{f}, \hat{g}]] = 0, \tag{3}$$

due to (1) and (2). So, the associativity is essential for the consistency of the canonical quantum mechanics.

However, some quantum mechanical systems are formulated in terms of non-associative algebras of the canonical operators. The standard example of such a situation is the introduction of the magnetic charges through the commutator of the covariant momenta, $[\hat{x}^i, \hat{x}^j] = 0$, $[\hat{x}^i, \hat{\pi}_j] = i\delta^i_j$ and $[\hat{\pi}_i, \hat{\pi}_j] = ie\varepsilon_{ijk}B^k(\hat{x})$, with $div\vec{B} \neq 0$, see e.g., [1,2] for more details. For the Dirac monopole, in particular, one has $\vec{B}(\vec{x}) = g\vec{x}/x^3$, with g being the magnetic charge. The Jacobi identity is violated only in one point (the position of the magnetic charge). To overturn this difficulty one may impose the appropriate boundary condition for the wave function leading to the famous Dirac quantization rule: $eg/2\pi\hbar \in \mathbb{Z}$. For the linear magnetic field, $\vec{B} = g\vec{x}/3$, the jacobiator is constant, meaning that one cannot repeat the same trick to fix the problem.

Another source of the examples of non-associative structures is the string theory. Recent advances in understanding flux compactifications of string theory have suggested that non-geometric frames are related to non-commutative and non-associative deformations of space–time geometry [3–7]. Since, these flux deformations of geometry are probed by closed strings, they have a much better potential for providing an effective target space description of quantum gravity than previous appearances of non-commutative geometry in string theory. To give an example of arising a non-geometric background let us consider the closed strings propagating in a three-torus endowed with a constant Neveu–Schwarz flux $H = dB$. Applying consecutive T-duality transformations along all three directions one obtains the relation between geometric and non-geometric fluxes: $H \to f \to Q \to R$. The Q-flux background is non-commutative but associative, while the purely non-geometric R-flux background is not only non-commutative, but also non-associative. The presence of non-vanishing three-form H-flux in string compactifications makes the closed strings coordinates non-commutative and non-associative in the analogy with the coordinates of the open string endpoints attached to a D-brane in a background B-field [8–12].