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Counting surface-kernel epimorphisms from a co-compact Fuchsian group to a cyclic group with motivations from string theory and QFT

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Abstract

Graphs embedded into surfaces have many important applications, in particular, in combinatorics, geometry, and physics. For example, ribbon graphs and their counting is of great interest in string theory and quantum field theory (QFT). Recently, Koch et al. (2013) [12] gave a refined formula for counting ribbon graphs and discussed its applications to several physics problems. An important factor in this formula is the number of surface-kernel epimorphisms from a co-compact Fuchsian group to a cyclic group. The aim of this paper is to give an explicit and practical formula for the number of such epimorphisms. As a consequence, we obtain an 'equivalent' form of Harvey's famous theorem on the cyclic groups of automorphisms of compact Riemann surfaces. Our main tool is an explicit formula for the number of solutions of restricted linear congruence recently proved by Bibak et al. using properties of Ramanujan sums and of the finite Fourier transform of arithmetic functions.

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1. Introduction

A *surface* is a compact oriented two-dimensional topological manifold. Roughly speaking, a surface is a space that 'locally' looks like the Euclidean plane. Informally, a graph is said to be *embedded into* (or *drawn on*) a surface if it can be drawn on the surface in such a way that its edges meet only at their endpoints. A *ribbon graph* is a finite and connected graph together with a cyclic ordering on the set of half edges incident to each vertex. One can see that ribbon graphs and embedded graphs are essentially equivalent concepts; that is, a ribbon graph can be thought as a set of disks (or vertices) attached to each other by thin stripes (or edges) glued to their boundaries. There are several other names for these graphs in the literature, for example, *fat graphs*, or *combinatorial maps*, or *unrooted maps*. For a thorough introduction to the theory of embedded graphs we refer the reader to the lovely book by Lando and Zvonkin [13].

Graphs embedded into surfaces have many important applications, in particular, in combinatorics, geometry, and physics. For example, ribbon graphs and their counting is of great interest in string theory and quantum field theory (QFT). Here we quote some of these applications and motivations from [11,12]:

- Ribbon graphs arise in the context of MHV rules for constructing amplitudes. In the MHV rules approach to amplitudes, inspired by twistor string theory, amplitudes are constructed by gluing MHV vertices. Counting ribbon graphs plays an important role here in finding different ways of gluing the vertices which contribute to a given amplitude.
- The number of ribbon graphs is the fundamental combinatorial element in perturbative large N QFT computations, since we need to be able to enumerate the graphs and then compute corresponding Feynman integrals.
- In matrix models (more specifically, the Gaussian Hermitian and complex matrix models), which can be viewed as QFTs in zero dimensions, the correlators are related very closely to the combinatorics of ribbon graphs. There is also a two-dimensional structure (related to string worldsheets) to this combinatorics.
- There is a bijection between vacuum graphs of Quantum Electrodynamics (QED) and ribbon graphs. In fact, the number of QED/Yukawa vacuum graphs with $2v$ vertices is equal to the number of ribbon graphs with v edges. This can be proved using permutations. Note that QED is an Abelian gauge theory with the symmetry circle group $U(1)$.

Mednykh and Nedela [18] obtained a formula for the number of unrooted maps of a given genus. Recently, Koch, Ramgoolam, and Wen [12] gave a refinement of that formula to make it more suitable for applications to several physics problems, like the ones mentioned above. In both formulas, there is an important factor, namely, the number of surface-kernel epimorphisms from a co-compact Fuchsian group to a cyclic group. A formula for the number of such epimorphisms is given in [18] but that formula does not seem to be very applicable, especially for large values, because one needs to find, as part of the formula, a challenging summation involving the products of some Ramanujan sums and for each index of summation one needs to calculate these products. The aim of this paper is to give a very explicit and practical formula for the number of such epimorphisms. Our formula does not contain Ramanujan sums or other challenging parts, and is really easy to work with. As a consequence, we obtain an 'equivalent' form of the famous Harvey's theorem on the cyclic groups of automorphisms of compact Riemann surfaces.

In the next section, we review Fuchsian groups and Harvey's theorem. Our main tool in this paper is an explicit formula for the number of solutions of restricted linear congruence recently