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Optimized Fock space in the large N limit of quartic interactions in matrix models

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Abstract

We consider the problem of quantization of the bosonic membrane via the large N limit of its matrix regularizations H_N in Fock space. We prove that there exists a choice of the Fock space frequency such that H_N can be written as a sum of a non-interacting Hamiltonian $H_{0,N}$ and the original normal ordered quartic potential. Using this decomposition we obtain upper and lower bounds for the ground state energy in the planar limit, we study a perturbative expansion about the spectrum of $H_{0,N}$, and show that the spectral gap remains finite at $N = \infty$ at least up to the second order. We also apply the method to the $U(N)$-invariant anharmonic oscillator, and demonstrate that our bounds agree with the exact result of Brezin et al.

1. Introduction

Let us consider the classical internal energy of the bosonic membrane, which in a light-cone description in orthonormal gauge can be written as (for more details see e.g. [1])

$$\mathbb{M}^2 = \int_{\Sigma} \left(\frac{\vec{p}^2}{\rho} + \rho \sum_{i<j} \{x_i, x_j\} \right) d^2\varphi, \tag{1.1}$$

with the constraints

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$$\sum_{i=1}^{d}\{x_i, p_i\} = 0, \tag{1.2}$$

where the integral is performed over a 2-dimensional compact manifold Σ and $\{f, g\} := \frac{1}{\rho(\varphi)}(\partial_1 f \partial_2 g - \partial_2 f \partial_1 g)$ denotes the Poisson bracket. It is convenient to use the mode expansions $x_i(\varphi) = x_{i\alpha} Y_\alpha(\varphi)$, $p_j(\varphi) = p_{j\alpha} Y_\alpha(\varphi)$ in terms of the eigenfunctions $\{Y_\alpha\}_{\alpha=1}^{\infty}$ of the Laplace operator on Σ, where the zero modes are subtracted. This allows to rewrite $\mathbb{M}^2$ as an infinite sum over the internal modes

$$\mathbb{M}^2 = p_{i\alpha} p_{i\alpha} + \frac{1}{2} g_{\alpha\beta\gamma} g_{\alpha\beta'\gamma'} x_{i\beta} x_{i\beta'} x_{j\gamma} x_{j\gamma'}, \tag{1.3}$$

$$g_{\alpha\beta\gamma} := \int Y_\beta \epsilon^{ab} \partial_a Y_\alpha \partial_b Y_\gamma d^2\varphi, \quad i = 1, \ldots, d, \quad \alpha, \beta, \gamma = 1, \ldots, \infty. \tag{1.4}$$

It has been shown by Goldstone and Hoppe [2,3] that the full field-theoretic Hamiltonian (1.3) admits a regularization procedure, where the classical phase-space variables $x_i(\varphi)$, $p_j(\varphi)$ are replaced by n-dimensional matrices, the Poisson bracket by the matrix commutator and integrals over Σ by the matrix trace. The original volume-preserving diffeomorphisms symmetry of Σ, represented by (1.2), is recovered in the $n \to \infty$ limit from the $SU(n)$ invariance of its matrix regularizations. The family of n-dimensional matrix models constructed in this way reads

$$H_N = Tr(\vec{P}^2) - (2\pi n)^2 n \sum_{i<j}^{d} Tr([X_i, X_j]^2), \tag{1.5}$$

with the $SU(n)$ invariance constraints

$$\sum_{i=1}^{d}[X_i, P_i] = 0, \tag{1.6}$$

where P_i, X_i are hermitian traceless $n \times n$ matrices. The scaling factor in front of the quartic potential is chosen in such a way that $\lim_{N\to\infty} H_N = \mathbb{M}^2$.

Using a basis of $su(n)$, T_a, $a = 1, \ldots, n^2 - 1 := N$, with $Tr(T_a T_b) = \delta_{ab}$ and $[T_a, T_b] = i\hbar_n \frac{1}{\sqrt{n}} f_{abc}^{(n)} T_c$, $\hbar_n = \frac{1}{2\pi n}$, $f_{abc}^{(n)} = \frac{2\pi n^{\frac{3}{2}}}{i} Tr(T_a [T_b, T_c])$, we can rewrite H_N (and the constraints) in terms of $d(n^2 - 1)$ canonical pairs p_{ia}, x_{ia} ($X_i = x_{ia} T_a$, $P_i = p_{ia} T_a$) as a *finite* sum over the matrix modes, cp. (1.3),

$$H_N(p, x) = p_{ia} p_{ia} + \frac{1}{2} f_{abc}^{(n)} f_{ab'c'}^{(n)} x_{ib} x_{ib'} x_{jc} x_{jc'}, \tag{1.7}$$

$$f_{abc}^{(n)} x_{ib} p_{jc} = 0. \tag{1.8}$$

In contrast to string theory, the Hamiltonian of the membrane with its quartic interaction makes the problem of quantization rather difficult. One approach, which was proposed in the literature many years ago [2,3], is to take advantage of the symmetry preserving matrix regularizations (1.5)/(1.7), quantize it for finite N and then take the limit $N \to \infty$. While it has been proved that all finite N expressions are well defined Schrödinger operators[1] with purely discrete spectrum [4,5], it seems that almost nothing is known about the large N limit (apart from the

[1] See [6] for a discussion of the spectrum, including the supersymmetric version of the model and related issues.