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Bridges in the random-cluster model

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Abstract

The random-cluster model, a correlated bond percolation model, unifies a range of important models of statistical mechanics in one description, including independent bond percolation, the Potts model and uniform spanning trees. By introducing a classification of edges based on their relevance to the connectivity we study the stability of clusters in this model. We prove several exact relations for general graphs that allow us to derive unambiguously the finite-size scaling behavior of the density of bridges and non-bridges. For percolation, we are also able to characterize the point for which clusters become maximally fragile and show that it is connected to the concept of the bridge load. Combining our exact treatment with further results from conformal field theory, we uncover a surprising behavior of the (normalized) *variance* of the number of (non-)bridges, showing that it diverges in two dimensions below the value $4\cos^2(\pi/\sqrt{3}) = 0.2315891\cdots$ of the cluster coupling q. Finally, we show that a partial or complete pruning of bridges from clusters enables estimates of the backbone fractal dimension that are much less encumbered by finite-size corrections than more conventional approaches.

1. Introduction

Percolation is probably the most widely discussed and arguably the simplest model of critical phenomena. Due to a combination of conceptual simplicity and wide applicability which is a

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signature of a problem of very general interest, its many incarnations including bond and site percolation on a lattice as well as continuum and non-equilibrium, directed variants, have been the subject of thousands of studies [1]. In statistical physics and mathematics alike, the quest to understand aspects of the percolation problem has led to developments of powerful and beautiful new techniques [2,3]. While the problem is well defined and interesting for any graph or lattice, both finite and infinite, the understanding of non-trivial cases is most advanced in two dimensions. There, the scaling limit of critical percolation can be related to the Coulomb gas [4] and conformal field theory [5], leading to exact results for most critical exponents and certain correlation functions. More recently, a rigorous approach to conformal field theory was pioneered by Schramm who used a mapping introduced by Löwner to construct a way of generating conformally invariant fractal random curves, the Stochastic (or Schramm) Löwner Evolutions (SLEs) [6], for which a range of properties, including fractal dimensions, can be calculated exactly. In this context, Smirnov and co-workers used the concept of discrete analyticity to establish rigorously that the scaling limits of critical percolation [7,8] and the Ising model [9] on the triangular lattice are indeed conformally invariant, and cluster boundaries in these models converge to certain classes of SLE traces.

The random-cluster (RC) model was suggested by Fortuin and Kasteleyn as a natural extension of the (bond) percolation problem, noting that there was a class of models fulfilling the series and parallel laws of electrical circuits that also included the Ising model [10]. Given a graph $G = (V, E)$, it assigns to a spanning subgraph $(V, A \subseteq E)$ a probability mass (in the following also referred to as RC measure) [11]

$$\mathbb{P}_{p,q,G}[A] = \frac{p^{|A|}(1-p)^{|E|-|A|}q^{K(A)}}{Z_{\mathrm{RC}}(p,q,G)}, \quad A \in \Omega_G, \tag{1.1}$$

where $K(A)$ is the number of components and $|A|$ the number of edges in A. The quantity $Z_{\mathrm{RC}}(p,q,G)$ is the partition function of the RC model, corresponding to the sum of unnormalized weights, and Ω_G constitutes the set of all spanning subgraphs or configurations, i.e., $\Omega_G = \{A : A \subseteq E\}$. Edges that are in A are called **open** and those in $E \setminus A$ **closed**. The presence of the cluster-weight factor $q^{K(A)}$ distinguishes (1.1) from the percolation problem and prevents $\mathbb{P}_{p,q,G}[A]$ from being a Bernoulli product measure; only for $q \to 1$, where the model reduces to the percolation problem and hence edges become independent, this property is restored. Although the cluster weight q can be any non-negative real number, integer values of q are particular in that for them the partition function is very closely related to that of the q-state Potts model [12] (see Eq. (2.3) below). For lattice graphs in at least two dimensions, the model undergoes a percolation phase transition at a critical value $p_c(q)$ of the bond probability where, for sufficiently large q, the transition becomes discontinuous [11]. On the square lattice, self-duality allows to deduce the exact transition point $p_{\mathrm{c}}(q) = p_{\mathrm{sd}}(q) = \sqrt{q}/(1+\sqrt{q})$ [13], and the location of the tricritical point is known to be $q_{\mathrm{c}} = 4$, beyond which the transition becomes of first order [14].

The crucial importance of the RC description for the understanding of critical phenomena is through its expression in purely geometrical terms. Hence understanding the geometric structure of the (correlated) percolation problem (1.1) provides a geometric route to the understanding of the thermal phase transition of the Potts model. Correspondingly, significant effort has been devoted in particular to investigations of the structure of the incipient percolating cluster. While initially it was assumed that it was a network of *nodes* connected by essentially one-dimensional *links*, results regarding the conductivity of the critical cluster implied that, instead, the structure is better described by the more elaborate ‘links–nodes–blobs’ picture [15]. If one fixes two distant points A and B on the cluster, those bonds that have independent, non-intersecting paths to