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Mean distribution approach to spin and gauge theories

Oscar Akerlund [a,*], Philippe de Forcrand [a,b]

[a] *Institut für Theoretische Physik, ETH Zürich, CH-8093 Zürich, Switzerland*
[b] *CERN, Physics Department, TH Unit, CH-1211 Geneva 23, Switzerland*

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Abstract

We formulate self-consistency equations for the distribution of links in spin models and of plaquettes in gauge theories. This improves upon known mean-field, mean-link, and mean-plaquette approximations in such that we self-consistently determine all moments of the considered variable instead of just the first. We give examples in both Abelian and non-Abelian cases.

1. Introduction

It is always of interest to think about methods that allow easy extraction of approximate results, even though the computer power available for exact simulations is growing at an ever increasing pace. Mean-field methods are often qualitatively reliable in their self-consistent determination of the long-distance physics, and have a wide range of applications, with spin models as typical examples. For a gauge theory, formulated in terms of the gauge links, however, it is questionable what a *mean link* would mean, because of the local nature of the symmetry. This can be addressed by fixing the gauge, but the mean-field solution will then in general depend on the gauge-fixing parameter. Nevertheless, Drouffe and Zuber developed techniques for a mean field treatment of general Lattice Gauge Theories in [1] and showed that for fixed βd, where β is the inverse gauge coupling and d the dimension, the mean-field approximation can be consid-

* Corresponding author.
E-mail addresses: oscara@itp.phys.ethz.ch (O. Akerlund), forcrand@itp.phys.ethz.ch (P. de Forcrand).

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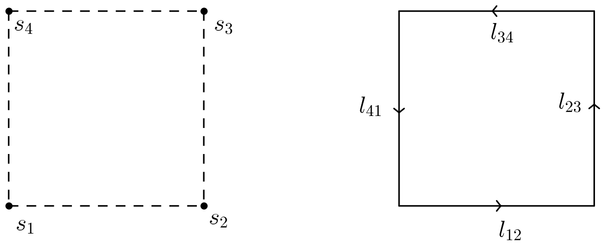

Fig. 1. The change of variables from spins s_i (*left panel*) to links l_{ij} (*right panel*) that leads to the Bianchi identity $l_{12}l_{23}l_{34}l_{41}(= s_1 s_2^\dagger s_2 s_3^\dagger s_3 s_4^\dagger s_4 s_1^\dagger) = 1$.

ered the first term in a $1/d$ expansion. They established that the mean field approximation can be thought of as a resummation of the weak coupling expansion in a particular gauge and that there is a first order transition to a strong coupling phase at a critical value of β. Since it becomes exact in the $d \to \infty$ limit, this mean field approximation can be used with some confidence in high-dimensional models [2].

The crucial problem of gauge invariance was tackled and solved by Batrouni in a series of papers [3,4], where he first changed variables from gauge-variant links to gauge-invariant plaquettes. The associated Jacobian is a product of lattice Bianchi identities, which enforce that the product of the plaquette variables around an elementary cube is the identity element. In the Abelian case this is easily understood, since each link occurs twice (in opposite directions) and cancels in this product, leaving the identity element. In the non-Abelian case the plaquettes in each cube have to be parallel transported to a common reference point in order for the cancellation to work. It is worth noting that in two dimensions there are no cubes so the Jacobian of the transformation is trivial and the new degrees of freedom completely decouple (up to global constraints).

This kind of change of variables can be performed for any gauge or spin model whose variables are elements of some group. Apart from gauge theories, examples include $\mathbb{Z}_N$-spin models, $O(2)$- and $O(4)$-spin models and matrix-valued spin models. In spin models, the change of variables is from spins to links and the Bianchi constraint dictates that the product of the links around an elementary plaquette is the identity element. A visualization of the transformation and the Bianchi constraint for a $2d$ spin model is given in Fig. 1.

Let us review the change of variables for a gauge theory [4]. The original variables are links. The new ones are plaquettes. Under the action of the original symmetry of the model, the new variables transform within equivalence classes and it is possible to employ a mean field analysis to determine the "mean equivalence class". As usual we first choose a set of *live* variables, which keep their original dynamics and interact with an external bath of mean-valued fields. Interactions are generated through the Jacobian, which is a product of Bianchi identities represented by δ-functions

$$\delta\left(\prod_{P\in\partial C} U_P - 1\right), \tag{1}$$

where P denotes a plaquette and ∂C denotes the oriented boundary of the elementary cube C. The δ-functions can be represented by a character expansion in which we can replace the characters at the external sites by their expectation, or mean, values. Upon truncating the number of representations, this yields a closed set of equations in the expectation values which can be