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Bethe's quantum numbers and rigged configurations

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Abstract

We propose a method to determine the quantum numbers, which we call the rigged configurations, for the solutions to the Bethe ansatz equations for the spin-$1/2$ isotropic Heisenberg model under the periodic boundary condition. Our method is based on the observation that the sums of Bethe's quantum numbers within each string behave particularly nicely. We confirm our procedure for all solutions for length 12 chain (totally 923 solutions).

1. Introduction

Bethe's seminal solution to the isotropic Heisenberg model under the periodic boundary condition [3] in 1931 is one of the prototypical theories on the quantum integrable systems. Basic procedure of the algebraic Bethe ansatz is as follows (see, e.g., [7,14]). First, we solve the set of algebraic equations called the Bethe ansatz equations (see equation (12) in the main text). Next, by using the solutions to the Bethe ansatz equations, we construct eigenstates of the Hamiltonian (see equation (11)). The main problem which we will consider in the present paper is to show

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that the whole procedure involved in the Bethe ansatz method is mathematically well-defined. We pursue the problem in the same setting as was treated by Bethe.

For a long time it has been observed that there are several subtle points about the procedure. One of such obstructions is the problem called the singular solutions (see equation (13)). Recently we have much progress on this issue (see [1,2,16,10,12,13]) and now we attain fairly good understanding of the problem. However, the very structure of the solutions to the Bethe ansatz equations itself still remains in rather foggy situation.

In the analysis of the solutions to the Bethe ansatz equations, it has been customary to assume that roots of the solutions take a particular form called the strings. This idea was already apparent in the original Bethe's paper. However, as we will explain shortly after, the notion "string" has rather elusive nature and for a long time it has been a difficult problem to understand how to utilize the string structure in a proper manner. One of the well-known attempts on this problem is Takahashi's theory [20]. Let us explain in more detail. Let N be the length of the spin chain and let ℓ be the number of down spins. Then a main assumption in the derivation of Takahashi's quantum number is to suppose that the strings take exactly the following form[1]:

$$a+bi,\, a+(b-1)i,\, a+(b-2)i,\, \ldots,\, a-bi, \qquad (a\in\mathbb{R},\, b\in\mathbb{Z}_{\geq 0}/2). \tag{1}$$

However, when $\ell \geq 3$, the assumption (1) has serious difficulty. Indeed, as we can easily see in examples, the real part a as well as the intervals between successive roots are not unique if the lengths of the strings are larger than 2. As the result, Takahashi's quantum numbers are not uniquely defined and also they are not half-integers. In a previous paper [19], we proposed to seek an alternative to Takahashi's quantum numbers. In particular, in that paper we showed that the correct quantum number for the exceptional real solution [6] is different from the one derived by Takahashi's quantum numbers.

In the present paper, as a continuation of a previous work [19], we propose a method to assign quantum numbers to strings of roots. For this purpose, we start from the so-called Bethe's quantum numbers (see Section 3). Here, roughly speaking, Bethe's quantum numbers appear as phase factors after taking logarithm of the Bethe ansatz equations. By definition, Bethe's quantum numbers are uniquely defined and exactly integers or half-integers. Then our basic observation is that the *sum of Bethe's quantum numbers* associated with all roots of a given string behaves in a particularly simple manner. This is a remarkable property since individual Bethe's quantum numbers behave in a rather complicated way. Based on this observation we propose a method to determine complete set of the quantum numbers, which we call the rigged configurations, from Bethe's quantum numbers. We confirmed these observations for all solutions of $N=12$ case by using the numerical data given in [8,10].

As we noted previously, we do not have thorough understanding of the string pattern which appear in the solutions to the Bethe ansatz equations. However, our main aim in the present paper is to propose a method which could be made mathematically rigorous once we obtain sound understanding of the string structure. Our expectation relies on the fact that Bethe's numbers are uniquely defined and exactly half-integers.

To our best knowledge, Bethe's quantum numbers had been introduced and studied in the original paper by Bethe [3], formulas (37a) and (37b), and has been studied more recently in

[1] We want to point out that the assumption of the validity of the string shape (1) allows to guess some explicit expressions for q-weight multiplicities (Kostka–Foulkas polynomials) appearing in the representation theory of Lie algebras of type A. All these formulas have been proven rigorously in [11] and lead to applications in combinatorics, representation theory and discrete integrable systems.