Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 905 (2016) 447–479

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Box graphs and resolutions I

Andreas P. Braun [a,*], Sakura Schäfer-Nameki [b]

[a] University of Oxford, Mathematical Institute, Andrew Wiles Building, Woodstock Rd., Oxford OX2 6GG, UK
[b] Department of Mathematics, King's College, The Strand, London WC2R 2LS, UK

Received 11 November 2015; received in revised form 29 January 2016; accepted 1 February 2016

Available online 8 February 2016

Editor: Stephan Stieberger

Abstract

Box graphs succinctly and comprehensively characterize singular fibers of elliptic fibrations in codimension two and three, as well as flop transitions connecting these, in terms of representation theoretic data. We develop a framework that provides a systematic map between a box graph and a crepant algebraic resolution of the singular elliptic fibration, thus allowing an explicit construction of the fibers from a singular Weierstrass or Tate model. The key tool is what we call a fiber face diagram, which shows the relevant information of a (partial) toric triangulation and allows the inclusion of more general algebraic blowups. We shown that each such diagram defines a sequence of weighted algebraic blowups, thus providing a realization of the fiber defined by the box graph in terms of an explicit resolution. We show this correspondence explicitly for the case of $SU(5)$ by providing a map between box graphs and fiber faces, and thereby a sequence of algebraic resolutions of the Tate model, which realizes each of the box graphs.

1. Introduction

Elliptic fibrations have a rich mathematical structure, which dating back to Kodaira and Néron's work [1,2] on the classification of singular fibers has been in close connection with the theory of Lie algebras. Recently, this connection was been extended with a representation-theoretic characterization of singular fibers in higher codimension, in particular for three- and

* Corresponding author.
E-mail address: braun@maths.ox.ac.uk (A.P. Braun).

http://dx.doi.org/10.1016/j.nuclphysb.2016.02.002

four-folds [3]. The inspiration for this work came from string theory and supersymmetric gauge theory, in particular the Coulomb branch phases of three-dimensional $N = 2$ gauge theories. However the final result can be entirely presented in terms of geometry and representations of Lie algebras overlayed with a combinatorial structure, the so-called box graphs. The purpose of this paper is to complement this description of singular elliptic fibers with a direct resolution of singularity approach, and to develop a systematic way to construct the resolutions based on their description in terms of box graphs.

Consider a singular elliptic fibration with two or three-dimensional base B. In codimension one, the singular fibers fall into the Kodaira–Néron classification, and for ADE type Lie algebras, the singular fibers are a collection of $\mathbb{P}^1$s intersecting in an affine ADE Dynkin diagram. The main interest in the present work and the motivation for the works [3,4] is the extension of this to higher codimension fibers. Consider a singular Weierstrass (or Tate) model

$$y^2 = x^3 + fx + g, \tag{1}$$

which describes the elliptic fibration. As is well known, the main advantage of this is that we do not need to specify the base, except for requiring that the sections f, g (or the corresponding sections of the Tate model) exist. Then the discriminant of this equation characterizes the loci in the case where the fiber becomes singular. Let $z = 0$ be the local description of a component of the discriminant $\Delta = 4f^3 + 27g^2$. I.e. Δ has an expansion $\Delta = \delta_0 z^{n_0} + \delta_1 z^{n_1} + \cdots$. The vanishing order in z of (f, g, Δ) determines the Kodaira type of the singular fiber above the codimension one locus $z = 0$. Along special codimension two loci, $z = \delta_0 = 0$, the vanishing order of the discriminant increases, and thereby the singularity type enhances.

The box graphs provide answers to the following questions: for a fixed codimension one Kodaira singular fiber, what are the possible fiber types that can arise in codimension two and three. Secondly, how many distinct such fibers in codimension two and three are there, and how are these related through flop transitions. The Kodaira classification can be thought of as associating a Lie algebra $\mathfrak{g}$ (or affine Dynkin diagram) to the codimension one fibers. In the same spirit, the box graph supplements this with codimension two information, which is encoded in the representation-theoretic data associated to $\mathfrak{g}$. More precisely, the box graphs are sign (or color) decorated representation graphs. They give a succinct and elegant answer to these questions by characterizing the possible higher codimension fibers in terms of representation theoretic data alone. The box graphs determine the extremal generators of the cone of effective curves in codimension two and three, and flop transitions are implemented in terms of simple operations on the graph.

Box graphs are applicable to all Kodaira fibers in codimension one [3] and provide a framework to classify the fibers in higher codimension. One of the most studied examples is the case of $\mathfrak{su}(5)$, largely due to its relevance in F-theory compactifications, but also because it is one of the simplest examples which contains various interesting features of codimension two and three fibers. In this case the flop diagram was determined [4] in the map to the Coulomb branch of the three-dimensional $N = 2$ gauge theory that describes low energy effective theory of the M-theory compactification on the resolved elliptic fibration [5–9] and confirmed from the box graphs in [3].

This simple description in terms of box graphs is in stark contrast to the process of explicitly constructing crepant resolutions of singular fibers for elliptic three- and four-folds. The starting point for this process is the singular Weierstrass or Tate model and the resolutions are either