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Nuclear Physics B 905 (2016) 480–530

NUCLEAR PHYSICS B

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Box graphs and resolutions II: From Coulomb phases to fiber faces

Andreas P. Braun [a,*], Sakura Schäfer-Nameki [b]

[a] *University of Oxford, Mathematical Institute, Andrew Wiles Building, Woodstock Rd, Oxford OX2 6GG, UK*
[b] *Department of Mathematics, King's College, The Strand, London WC2R 2LS, UK*

Received 11 November 2015; received in revised form 29 January 2016; accepted 1 February 2016

Available online 10 February 2016

Editor: Stephan Stieberger

Abstract

Box graphs, or equivalently Coulomb phases of three-dimensional $N = 2$ supersymmetric gauge theories with matter, give a succinct, comprehensive and elegant characterization of crepant resolutions of singular elliptically fibered varieties. Furthermore, the box graphs predict that the phases are organized in terms of a network of flop transitions. The geometric construction of the resolutions associated to the phases is, however, a difficult problem. Here, we identify a correspondence between box graphs for the gauge algebras $\mathfrak{su}(2k + 1)$ with resolutions obtained using toric tops and generalizations thereof. Moreover, flop transitions between different such resolutions agree with those predicted by the box graphs. Our results thereby provide explicit realizations of the box graph resolutions.

1. Introduction

Beyond its applications in the modeling of particle physics and classification of 6d superconformal field theories, recent developments in F-theory have led to tremendous progress in uncovering properties of higher-dimensional elliptically fibered complex varieties. Much of the

* Corresponding author.
E-mail address: braun@maths.ox.ac.uk (A.P. Braun).

http://dx.doi.org/10.1016/j.nuclphysb.2016.02.001

progress has been made in particular in the study of crepant resolutions of singular elliptic fibers in higher dimensional varieties, i.e. resolutions that keep the canonical class unchanged.

The canonical setup of interest in F-theory compactifications [1] is an elliptically fibered Calabi–Yau variety in dimension 3 and 4, which models $N = (1, 0)$ six dimensional or four-dimensional $N = 1$ supersymmetric field theories, with gauge algebra $\mathfrak{g}$, and matter in the representations $\mathbf{R}_i$ of $\mathfrak{g}$. Four-folds in addition allow for codimension three singularities, where Yukawa couplings are realized. The F-theory limit is obtained by taking the volume of the fiber to zero, and this singular limit is in fact not sensitive to which crepant resolution is used [2,3]. However, various more refined aspects of the F-theory compactification, such determining the G_4-flux, the possible $U(1)$ and discrete symmetries, make use of the singularity resolutions.

By Kodaira's classification of singular fibers, one can associate a Lie algebra $\mathfrak{g}$ to an elliptic fibration. These are characterized in terms of an ADE type affine Dynkin diagram representing the dual graph to the intersection graph of the rational curves in the singular fiber. This classification holds for all singular fibers over codimension one loci in the base. In higher codimension, this classification ceases to be comprehensive, and additional structures emerge that are required in order to characterize how higher codimension singular fibers can occur, and what their characterization is.

In [4] (see also [5–9]), inspired by the correspondence to classical Coulomb phases in 3d and 5d supersymmetric gauge theories [5,10–14], a proposal was put forward to systematically describe the distinct small resolutions of singular elliptic fibrations, including fibers in codimension two and three. In addition to a Lie algebra $\mathfrak{g}$, which characterizes the codimension one fibers, the codimension two fibers have a representation $\mathbf{R}$ of $\mathfrak{g}$ associated to them, and by [4], the fibers can be obtained by a decorated representation graph, or box graph. Flops between distinct small resolutions are realized by the action of a quotiented Weyl group. Note that the box graphs are motivated from a dual M-theory compactification point of view and map the problem of small resolutions to Coulomb phases. However, as shown in [4], the analysis applies directly in the cone of effective curves of the elliptic fibration, and does not require any reference to the gauge theory. Recently, this work was utilized in [7] to determine a classification of the fibers in codimension two with additional $U(1)$ symmetries, which geometrically are realized in terms of rational sections. This has led to a survey of all F-theory Grand Unified Theories (GUTs) with additional $U(1)$ symmetries, with interesting phenomenological implications [15]. Thus the results on codimension two fibers are not merely of mathematical relevance, but indeed have far-reaching implications for the particle physics, in particular flavor properties, of F-theory compactifications.

Beyond this abstract characterization of elliptic fibrations, much progress has been made in the direct realization of elliptic curves in terms of hypersurfaces or complete intersections, for instance in toric varieties [16–23]. What is apparent from all these resolutions is that neither toric, nor algebraic resolutions necessarily yield the full set of possible fibers predicted in [4]. Concrete realizations of the complete set of distinct resolutions have indeed been determined for $\mathfrak{su}(5)$ in [5,6,9], with both fundamental and anti-symmetric matter, in terms of resolutions of the Tate model for a codimension one I_5 Kodaira fiber [24,25].

The purpose of the present work is to clarify the connection between toric and algebraic resolutions on the one hand, and the more general resolutions that are predicted by the box graphs, on the other. We will determine a characterization of all algebraic resolutions in terms of a subclass of box graphs, which have a simple combinatorial description. Furthermore, resolutions associated to triangulations of toric tops [26] are determined in terms of triangulations of a so-called *fiber face*. We then show how fiber face triangulations form a subset of the box graph resolutions