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Off-diagonal Bethe ansatz solutions of the anisotropic spin-$\frac{1}{2}$ chains with arbitrary boundary fields

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Abstract

The anisotropic spin-$\frac{1}{2}$ chains with arbitrary boundary fields are diagonalized with the off-diagonal Bethe ansatz method. Based on the properties of the R-matrix and the K-matrices, an operator product identity of the transfer matrix is constructed at some special points of the spectral parameter. Combining with the asymptotic behavior (for XXZ case) or the quasi-periodicity properties (for XYZ case) of the transfer matrix, the extended $T-Q$ ansatzes and the corresponding Bethe ansatz equations are derived.

Keywords: Spin chain; Reflection equation; Bethe ansatz; $T-Q$ relation

1. Introduction

The study of exactly solvable models (or quantum integrable systems) [1] has attracted a great deal of interest since Yang and Baxter's pioneering works [2,3]. Such exact non-perturbation results have provided valuable insight into the important universality classes of quantum physical systems ranging from modern condensed matter physics [4] to string and super-symmetric Yang–Mills theories [5–8]. Moreover, quantum integrable models are paramount for the analysis of nano-scale systems where alternative approaches involving mean field approximations or perturbations have failed [9,10].

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The quantum inverse scattering method [11] (QISM) or the algebraic Bethe ansatz method has been proven to be the most powerful and (probably) unified method to construct exact solutions to the spectrum problem of commuting families of conserved charges (usually called the transfer matrix) in quantum integrable systems. In the framework of QISM, the quantum Yang–Baxter equation (QYBE), which defines the underlying algebraic structure, has become a cornerstone for constructing and solving the integrable models. So far, there have been several well-known methods for deriving the Bethe ansatz (BA) solutions of integrable models: the coordinate BA [12–15], the $T-Q$ approach [1,16], the algebraic BA [17,18,11,19], the analytic BA [20], the functional BA [21] or the separation of variables method among many others [22–37].

Generally, there are two classes of integrable models: one possesses $U(1)$ symmetry and the other does not. Three well-known examples without $U(1)$ symmetry are the XYZ spin chain [3,18], the spin chains with antiperiodic boundary condition [38,39,33–35,37] and the ones with unparallel boundary fields [24–27,32–37]. It has been proven that most of the conventional Bethe ansatz methods can successfully diagonalize the integrable models with $U(1)$ symmetry. However, for those without $U(1)$ symmetry, only some very special cases such as the XYZ spin chain with even site number [3,18] and the XXZ spin chain with constrained unparallel boundary fields [19,25,26,40] can be dealt with due to the existence of a proper "local vacuum state" in these special cases. The main obstacle for applying the algebraic Bethe ansatz and Baxter's method to general integrable models without $U(1)$ symmetry lies in the absence of such a "local vacuum". A promising method for approaching such kind of problems is Sklyanin's separation of variables method [21] which has been recently applied to some integrable models [34–37]. However, before the very recent work [41], a systematic method was absent to derive the Bethe ansatz equations for integrable models without $U(1)$ symmetry, which are crucial for studying the physical properties in the thermodynamic limit.

As for integrable models without $U(1)$ symmetry, the transfer matrix contains not only the diagonal elements but also some off-diagonal elements of the monodromy matrix. This breaks down the usual $U(1)$ symmetry. Very recently, a systematic method [41] for dealing with such kind of models was proposed by the present authors, which has been shown [42] to successfully construct the exact solutions of the open XXX chain with unparallel boundary fields and the closed XYZ chain with odd site number. The central idea of the method is to construct a proper $T-Q$ ansatz with an extra off-diagonal term (comparing with the ordinary ones) based on the functional relations between eigenvalues $\Lambda(u)$ of the transfer matrix (the trace of the monodromy matrix) and the quantum determinant $\Delta_q(u)$, (e.g. see below (4.9) and (5.22)) at some special points of the spectral parameter $u=\theta_j$, i.e.,

$$\Lambda(\theta_j)\Lambda(\theta_j-\eta)\sim\Delta_q(\theta_j). \tag{1.1}$$

Since the trace and the determinant are two basic quantities of a matrix which are independent of the representation basis, this method could overcome the obstacle of absence of a reference state which is crucial in the conventional Bethe ansatz methods. Moreover, in this paper we will show that the above relation can be lifted to operator level (see below (3.8)) based on some properties of the R-matrix and K-matrices.

Our primary motivation for this work comes from the longstanding problem of solving the anisotropic spin-$\frac{1}{2}$ chain with arbitrary boundary fields, defined by the Hamiltonian [43,44]

$$H=\sum_{j=1}^{N-1}\left(J_x\sigma_j^x\sigma_{j+1}^x+J_y\sigma_j^y\sigma_{j+1}^y+J_z\sigma_j^z\sigma_{j+1}^z\right)+\vec{h}^{(-)}\cdot\vec{\sigma}_1+\vec{h}^{(+)}\cdot\vec{\sigma}_N, \tag{1.2}$$