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Integration by parts identities in integer numbers of dimensions. A criterion for decoupling systems of differential equations

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Abstract

Integration by parts identities (IBPs) can be used to express large numbers of apparently different d-dimensional Feynman Integrals in terms of a small subset of so-called master integrals (MIs). Using the IBPs one can moreover show that the MIs fulfil linear systems of coupled differential equations in the external invariants. With the increase in number of loops and external legs, one is left in general with an increasing number of MIs and consequently also with an increasing number of coupled differential equations, which can turn out to be very difficult to solve. In this paper we show how studying the IBPs in fixed integer numbers of dimension $d = n$ with $n \in \mathbb{N}$ one can extract the information useful to determine a new basis of MIs, whose differential equations decouple as $d \to n$ and can therefore be more easily solved as Laurent expansion in $(d - n)$.

1. Introduction

Dimensionally regularised [1–3] Feynman integrals fulfil different identities, among which the most notable ones are the so-called integration by parts identities (IBPs) [4,5]. Given a family of Feynman integrals, the IBPs can be used to write down a large system of linear equations

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with rational coefficients, which contain the Feynman integrals of that family as unknowns. By solving algebraically the system a large number of apparently different Feynman integrals can be expressed in terms of a much smaller basis of independent integrals dubbed master integrals (MIs). In realistic applications the number of such equations can grow very fast, requiring the use of computer algebra in order to handle the complexity of the resulting expressions. There are different public and private implementations which allow to perform the reduction to MIs in a completely automated way [6–9] based on the so-called Laporta algorithm [10,11].

The IBPs can be used to prove that dimensionally regularised Feynman integrals fulfil linear systems of first order differential equations in the external invariants [12–16]. A thorough review of the method can be found in [17]. Considering a Feynman graph which is reduced to N independent MIs, by direct use of the IBPs one can derive a system of N coupled linear first order differential equations for the latter, which can be rephrased as an N-th order differential equation for any of the MIs. Supplemented with N independent boundary conditions, the system of differential equations contains all information needed for numerical or analytical calculations of the MIs. Indeed, in the general case, the analytical solution of an N-th order differential equation is a very non-trivial mathematical problem.

It has been observed that, in many cases of practical interest, a substantial simplification of the problem occurs when studying the behaviour of the system of differential equations as the space–time dimension parameter d approaches 4, which is also the physically relevant case. Usually we are indeed not interested in an exact solution for the MIs as functions of d, but instead in the coefficients of their Laurent expansion for $d \approx 4$. In [18,19], and in many subsequent applications of the differential equation method, it was shown that it is often possible to choose a basis of MIs such that the differential equations take a simpler triangular form in the limit $d \to 4$. If this is possible, the problem of integrating the system of differential equations simplifies substantially, reducing *de facto*, at every order in $(d-4)$, to N subsequent integrations by quadrature. Experience showed that, whenever such a form is achievable, the differential equations can be integrated in terms of a particular class of special functions, the multiple polylogarithms (MPLs) [18,20,21]. The latter have been studied extensively by both mathematicians and physicists and routines for their fast and precise numerical evaluation are available since some time [22–24]. Disclosing their algebraic properties allowed furthermore the development of very powerful tools for the analytical manipulations of these functions [25–27].

More recently it has been shown [28–30] that in many of these cases a basis of MIs can be found, such that the system of differential equations takes a particularly simple form, commonly referred to as *canonical form*. The system is said to be in canonical form if the regularisation parameter d can be completely factorised from the kinematics, appearing as an explicit $(d-4)$ factor in front of the matrix of the system. In addition, the coefficients of the matrix must be total differentials of logarithms of functions of the external invariants (i.e. they are said to be in d-log form). A canonical basis is particularly convenient as it allows a straightforward integration as series expansion in $(d-4)$ and, due to the d-log form of the coefficients, it integrates directly to MPLs of uniform transcendental weight. Criteria for the construction of candidate canonical integrals have been presented in [29] and developed in detail, for example, in [31]. In the special cases in which the differential equations depend only linearly on the dimensions d, the Magnus algorithm can be used to perform a rotation of the system to a canonical form [32]. For a recent application of the algorithm see [33]. A completely different approach based on Moser algorithm [34] has been developed for the univariate case in [35] and discussed also independently in [36]. Another interesting approach is based on the properties of higher order differential equations fulfilled by the individual master integrals [37]. In spite of all this impressive progress,