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Finite size spectrum of $SU(N)$ principal chiral field from discrete Hirota dynamics

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Abstract

Using recently proposed method of discrete Hirota dynamics for integrable $(1+1)$D quantum field theories on a finite space circle of length L we derive and test numerically a finite system of nonlinear integral equations for the exact spectrum of energies of $SU(N) \times SU(N)$ principal chiral field model as functions of mL, where m is the mass scale. We propose a determinant solution of the underlying Y-system, or Hirota equation, in terms of Wronskian determinants of $N \times N$ matrices parameterized by $N-1$ functions of the spectral parameter θ with the known analytic properties at finite L. Although the method works in principle for any state, the explicit equations are written for states in the $U(1)$ sector only. For $N > 2$, we encounter and clarify a few subtleties in these equations related to the presence of bound states, absent in the previously considered $N = 2$ case. As a demonstration of efficiency of our method, we solve these equations numerically for a few low-lying states at $N = 3$ in a wide range of mL.

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1. Introduction

Integrable $1+1$ dimensional quantum field theories on a finite space circle have been rather intensively studied in the last 20 years [1–10]. A great deal of success in the exact treatment of the finite size effects in various integrable QFT's was due to the thermodynamic Bethe ansatz (TBA) approach [11] resulting in a system (in most of the cases infinite) of non-linear integral equations. It was realized that the TBA equations could be rewritten in a functional, Y-system form [2].

Recently, a novel, quite general approach to these problems was proposed in [12] based on the integrability of the Y-system. The Y-system is known to be a gauge invariant version of the famous Hirota bilinear equation, often called the T-system, in its discrete form [13]. The underlying discrete Hirota dynamics is integrable and general solutions of Hirota equations can be found in a determinant (Wronskian) form [14] for various boundary conditions corresponding to a variety of different problems, from matrix models to quantum spin chains and quantum sigma-models. For finite rank symmetry groups, the Wronskians contains only a finite number of functions of the spectral parameter. Thus the Wronskian solution can drastically simplify the problem: the infinite Y-system is reduced to a finite number of non-linear integral equations for these functions. Then the subtlest point comes: We should guess the analytic properties of these functions w.r.t. the spectral parameter. This is relatively easy to do for the spin chains where the polynomiality of transfer matrix leads to the final answer in terms of a set of Bethe ansatz equations. For the QFT's at a finite volume L (length of the space circle) the situation is much more complicated and the analyticity properties of the Y-functions are not so obvious. Nevertheless, it often appears to be possible to extract them, partially from physical considerations, partially from certain assumptions of the absence of unphysical singularities. It helps to transform the Y-system into a system of non-linear integral equations (NLIE), more tractable, and better suitable for the numerical studies. The resulting equations can remind the Destri–De Vega NLIE [1] or even coincide with them for a limited set of 2D QFT's where these DDV equations are known.

This program was first performed in [12] for the $SU(2)_L \times SU(2)_R$ principal chiral field (PCF) for a general quantum state, and the numerical study of the finite size spectrum was successfully done for a variety of interesting states, from the vacuum and mass-gap to quite general states, in the so called $U(1)$ sector or even lying out of it (i.e. having excitations in left and right $SU(2)$ spin modes).

In this paper, we will construct within these lines the corresponding NLIE's for $SU(N) \times SU(N)$ PCF at any N. We use the Wronskian solution of [14] for the underlying Hirota equation in terms of determinants of $N \times N$ matrices and guess the correct analytic form of the functions entering the Wronskian. For the vacuum state, the asymptotic Bethe ansatz (ABA) based on the scattering theory and, strictly speaking, valid only for sufficiently large length L teaches us that there are no singularities on the physical strip of the rapidity plane, at least for not too small L's.[2]

For excited states there are certain poles entering the physical strip, and their qualitative structure can be guessed from the ABA. The explicit construction is done only for states in the $U(1)$ sector, but we sketch out the generalization to any state. We show how the exact S-matrix of the model (including the CDD factor) naturally emerges from this approach based on the Y-system by simple analyticity assumptions.

[2] This argument based on ABA cannot exclude a possibility that at a sufficiently small size, some extra singularities occur. However, our numerics give serious evidence that at least for $N = 3$ such extra singularities do not appear.