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Classical solutions of a flag manifold σ-model

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Abstract

We study a σ-model with target space the flag manifold $\frac{U(3)}{U(1)^3}$ and a nonzero Kalb–Ramond field, which is specified by a choice of integrable complex structure on the target space. We describe the classical solutions of the model for the case when the worldsheet is a sphere $\mathbb{CP}^1$.

In his seminal paper [1] Pohlmeyer discovered that the σ-model with target space S^2 is classically integrable. He showed that this can be related to the fact that the equations of motion (e.o.m.) of the model are equivalent to the flatness of a one-parametric family of connections. Soon afterwards it was realized that analogous properties are shared by σ-models with symmetric target spaces [2]. The case of non-symmetric target spaces, however, resisted analysis by these methods. In [3] the author proposed a model with a homogeneous but not symmetric target space, with the property that its e.o.m. may be rewritten as a flatness condition for a one-parametric family of connections.

In this paper we will solve the e.o.m. of the σ-model proposed in [3] (reviewed in Sec. 2) for the case when the worldsheet $\mathscr{M}$ is a sphere $\mathbb{CP}^1$. The target space of the model is the manifold of full flags in $\mathbb{C}^3$, which we will denote by $\mathcal{F}_3$. It can be viewed as the space of ordered triples

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of orthogonal lines in $\mathbb{C}^3$ passing through the origin, and is also representable as a quotient space:

$$\mathcal{F}_3 = \frac{U(3)}{U(1)^3}\,. \tag{1}$$

The flag manifold $\mathcal{F}_3$ may be parametrized by the orthonormalized vectors u_i ($u_i \circ \bar{u}_j = \delta_{ij}$), modulo phase rotations $u_k \to e^{i\alpha_k} u_k$. Each of these vectors defines a point in projective space $\mathbb{CP}^2$, allowing to construct three natural forgetful maps $\{\pi_i : \mathcal{F}_3 \to \mathbb{CP}^2,\ i = 1, 2, 3\}$ by the formula $\pi_i(u_1, u_2, u_3) = u_i$. For this reason the properties of the flag manifold are tightly related to the properties of the underlying $\mathbb{CP}^2$'s. As we shall see, solutions to the flag σ-model e.o.m. are to a large extent expressible through the solutions of the $\mathbb{CP}^2$ model. Due to this, and to introduce the notation, we begin by defining the σ-model with target space $\mathbb{CP}^2$.

1. The $\mathbb{CP}^2$ σ-model

We will be thinking of $\mathbb{CP}^2$ as the quotient $\mathbb{CP}^2 = (\mathbb{C}^3 - \{0\})/\mathbb{C}^*$. A map $v : \mathscr{M} \to \mathbb{CP}^2$ from a Riemann surface $\mathscr{M}$ can be described by a vector-valued function $v(z, \bar{z}) \in \mathbb{C}^3$, where $z, \bar{z}$ are coordinates on the worldsheet $\mathscr{M}$. We may assume that the vector v is in fact normalized, that is $v \in S^5 \subset \mathbb{C}^3$: $\sum\limits_{i=1}^{3} |v_i|^2 := \bar{v} \circ v = 1$, and henceforth we will use this normalization. This is a partial gauge for the gauge group $\mathbb{C}^*$, which breaks it down to $U(1)$.

Introduce the covariant derivative

$$D_i^{(v)} w := \partial_i w - q_w \cdot (\bar{v} \circ \partial_i v)\, w\,, \qquad i = \{z, \bar{z}\} \tag{2}$$

where q_w is the $U(1)$-charge of w, normalized so that $q_v = 1$. In most of the applications of (2) below w is a vector obtained by applying covariant derivatives to the basic map v or its conjugate $\bar{v}$. For example, $w \in \{v, D_z^{(v)} v, D_{\bar{z}}^{(v)} v, D_z^{(v)} D_{\bar{z}}^{(v)} v, \ldots\}$, in which case $q_w = 1$, or $w \in \{\bar{v}, D_z^{(v)} \bar{v}, D_{\bar{z}}^{(v)} \bar{v}, D_z^{(v)} D_{\bar{z}}^{(v)} \bar{v}, \ldots\}$, in which case $q_w = -1$. When this does not lead to confusion, we will sometimes simply write D_z in place of $D_z^{(v)}$, $D_{\bar{z}}$ for $D_{\bar{z}}^{(v)}$.

The covariant derivative has the Leibniz property: $D_i^{(v)}(a \cdot b) = D_i^{(v)}(a) \cdot b + a \cdot D_i^{(v)}(b)$. The commutator of covariant derivatives produces the pull-back of the Fubini–Study form:

$$[D_z^{(v)}, D_{\bar{z}}^{(v)}] = D_{\bar{z}}^{(v)} \bar{v} \circ D_z^{(v)} v - D_z^{(v)} \bar{v} \circ D_{\bar{z}}^{(v)} v\,. \tag{3}$$

The action of the $\mathbb{CP}^2$ σ-model (with zero θ-term) is:

$$\mathcal{S} = \int\limits_{\mathscr{M}} \frac{i}{2}\, dz \wedge d\bar{z} \left(\|D_z v\|^2 + \|D_{\bar{z}} v\|^2 \right) \tag{4}$$

The equation of motion following from this action reads

$$D_{\bar{z}}^{(v)} D_z^{(v)} v = \alpha\, v, \tag{5}$$

where α is a scalar function. Multiplying this equation by $\bar{v}$ and using the Leibniz property of the covariant derivative together with the identity $\bar{v} \circ D_z^{(v)} v = 0$ (which follows from the