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Elliptically fibered Calabi–Yau manifolds and the ring of Jacobi forms

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Dedicated to Prof. S.-T. Yau on the occasion of his 65th Birthday

Abstract

We give evidence that the all genus amplitudes of topological string theory on compact elliptically fibered Calabi–Yau manifolds can be written in terms of meromorphic Jacobi forms whose weight grows linearly and whose index grows quadratically with the base degree. The denominators of these forms have a simple universal form with the property that the poles of the meromorphic form lie only at torsion points. The modular parameter corresponds to the fibre class while the role of the string coupling is played by the elliptic parameter. This leads to very strong all genus results on these geometries, which are checked against results from curve counting.

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1. Introduction

Determining the all genus topological string partition function

$$Z = \exp(F) = \exp\left(\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(\underline{z(\underline{t})})\right) \tag{1.1}$$

on a compact Calabi–Yau manifold M is a benchmark problem with many applications to enumerative geometry and physical implications for black holes and quantum supergravity. Here λ is the topological string coupling, t_a are complexified Kähler volumes associated to a basis β^a for $H_2(M,\mathbb{Z})$ and $q^\beta = \exp(2\pi i \sum_a t_a \beta^a)$. The underlining of variables is a shorthand for a corresponding list of indexed variables. The mirror map $z(\underline{t})$ can be determined from the solutions of the Picard–Fuchs equations describing the periods of the mirror manifold. At the large volume point $t_a \to i\infty$ and in the holomorphic limit $\lim_{\bar{\underline{t}}\to\infty} F_g = \mathcal{F}_g = \sum_{\beta\in H_2(M,\mathbb{Z})} r_g^\beta q^\beta$ one can read the genus g Gromov–Witten invariants r_g^β from the convergent $\underline{q}$ expansion of $\mathcal{F}_g$. In this holomorphic limit $\mathcal{Z} = \lim_{\bar{\underline{t}}\to\infty} Z$ also captures integer BPS invariants and related integer symplectic invariants, such as the Donaldson–Thomas and the Pandharipande–Thomas invariants. Up to the classical terms this limit is completely fixed by the unrefined BPS invariants $n_\beta^g \in \mathbb{Z}$ by the product formula

$$\mathcal{Z}(\underline{t},\lambda) = \prod_\beta \left[\left(\prod_{m=k}^{\infty} (1-y^k q^\beta)^{k n_0^\beta}\right) \prod_{g=1}^{\infty} \prod_{l=0}^{2g-2} (1-y^{g-l-1} q^\beta)^{(-1)^{g+l}\binom{2g-2}{l} n_\beta^g}\right], \tag{1.2}$$

where we defined $y = \exp(\lambda)$. This and similar product formulae have been developed following the pioneering work in this direction by Yau and Zaslow [17].

According to [2], for $g > 2$ the genus g topological string amplitudes $F_g(S^{ij}, S^j, S; \underline{z})$ are inhomogeneous polynomials of weighted degree $3g-3$ in the anholomorphic propagators S^{ij}, S^j, S, which have weights $(1,2,3)$ respectively. The coefficients of these polynomials are rational functions of $\underline{z}$ parametrizing the complex structure moduli space $\mathcal{M}_{cs}$ of M. The holomorphic anomaly of [2] determines all of F_g except the weight zero piece $f_g(\underline{z})$, whose determination is the key conceptual problem in the approach of [2]. Its solution requires information about the behavior of the modular invariant[1] nonholomorphic sections $F_g(\underline{z})$ over $\mathcal{M}_{cs}$ at all its boundary components. Following the method of [16] to obtain an efficient solution of the holomorphic anomaly equation, a careful analysis of the boundary behavior determined the f_g up to genus 51 for the quintic hypersurface in $\mathbb{P}^4$ [10], using the conifold gap condition, regularity at the orbifold point, and Castelnuovo's criterion for the vanishing of higher genus curves at large radius.

In this work we study whether the additional structures that arise when M has an elliptic fibration helps to fix the ambiguities f_g. We focus on a smooth elliptic fibration over the base $\mathbb{P}^2$ with a single holomorphic section, which has been considered in the context of mirror symmetry in [8,4]. It can be defined as a degree 18 hypersurface, i.e. a section of the anti-canonical class $P_{18} = 0$ in the resolved weighted projective space $\mathbb{P}^4_{1,1,1,6,9}$.

[1] By modular invariance we mean invariance under the monodromy group inside $\mathrm{SP}(6,\mathbb{Z})$ and discrete reparametrization symmetries. In our main example there is such a discrete involution symmetry, which implies $\mathrm{PSL}(2,\mathbb{Z})$ invariance on the modular parameter τ [9].