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An integrable nineteen vertex model lying on a hypersurface

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Abstract

We have found a family of solvable nineteen vertex model with statistical configurations invariant by the time reversal symmetry within a systematic study of the respective Yang–Baxter relation. The Boltzmann weights sit on a degree seven algebraic threefold which is shown birationally equivalent to the three-dimensional projective space. This permits to write parameterized expressions for both the transition operator and the R-matrix depending on three independent affine spectral parameters. The Hamiltonian limit tells us that the azimuthal magnetic field term is connected with the asymmetry among two types of spectral variables. The absence of magnetic field defines a physical submanifold whose geometrical properties are remarkably shown to be governed by a quartic K3 surface. This expands considerably the class of irrational manifolds that could emerge in the theory of quantum integrable models.

1. Introduction

At present the method of commuting transfer matrices provides the most important device for constructing exactly solvable lattice systems of statistical mechanics in two spatial dimensions [1]. Let us denote by $\mathrm{T_N}(\omega)$ the model transfer matrix defined on a given direction of the lattice with length N. In order to make notation simpler we have represented the lattice

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Boltzmann weights $\omega_1, \cdots, \omega_m$ by the single vector $\omega \in \mathbb{C}^m$. The commutativity of two transfer matrices with distinct weights implies the condition,

$$\left[\mathrm{T_N}\left(\omega'\right), \mathrm{T_N}\left(\omega''\right)\right] = 0, \quad \forall \omega' \text{ and } \omega'', \tag{1}$$

for arbitrary length N.

The fact that such commutation relation depends on the size N seems that one needs to verify an infinite number of relations among the Boltzmann weights to conclude that two different transfer matrices indeed commute. This is fortunately not the case since Baxter [1] argued that it is sufficient to solve only a finite set of algebraic relations to built up a family of commuting transfer matrices for any size N. This local condition is often referred to as the Yang–Baxter equation and its specific structure depends much on the class of lattice system under consideration. In this paper we are interested to investigate novel solutions to this relation in the case of lattice vertex models. We recall that the fluctuation variables of vertex models lie on the bonds between neighboring lattice points and the interaction energies depend on the allowed vertices configurations. The main feature of these models is their inherent tensor structure which allows us to construct the corresponding transfer matrices out of a single local transition operator. In the simplest case of rectangular lattices this operator acts on the direct product of the auxiliary and quantum spaces associated respectively to the horizontal and vertical edges statistical configurations. Assuming that each edge of the rectangular lattice can take values on q possible states one can represent the transition operator $\mathrm{L}(\omega)$ on the auxiliary space as the following $\mathrm{q} \times \mathrm{q}$ matrix,

$$\mathrm{L}(\omega) = \left(\begin{array}{c|c|c|c} \mathrm{W}_{1,1} & \mathrm{W}_{1,2} & \cdots & \mathrm{W}_{1,q} \\ \hline \mathrm{W}_{2,1} & \mathrm{W}_{2,2} & \cdots & \mathrm{W}_{2,q} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathrm{W}_{q,1} & \mathrm{W}_{q,2} & \cdots & \mathrm{W}_{q,q} \end{array}\right). \tag{2}$$

The entries $\mathrm{W}_{a,b}$ are also $\mathrm{q} \times \mathrm{q}$ operators but now acting on the space of quantum vertical degrees of freedom. Their matrix elements $\mathrm{W}_{a,b}(c,d)$ represent the Boltzmann weights for the edge horizontal states a, b and the edge vertical configurations c, d. The maximum number m of distinct Boltzmann weights the vertex model can have is therefore $m = \mathrm{q}^4$.

The transition operators can be combined to construct for instance the row-to-row transfer matrix represented as operators in the quantum space variables with an arbitrary number N of columns. Considering periodic boundary conditions on the horizontal direction the transfer matrix takes the form,

$$\mathrm{T_N}(\omega) = \mathrm{Tr_q}\left[\mathrm{L}_1(\omega)\mathrm{L}_2(\omega)\cdots\mathrm{L_N}(\omega)\right], \tag{3}$$

where the matrix multiplication and the trace operations are performed on the auxiliary space. The subscript index for the transition operator $\mathrm{L}_j(\omega)$ means that its matrix elements act non-trivially only at the j-th vertical quantum space of states.

A sufficient condition for the commutativity of $\mathrm{T_N}(\omega')$ and $\mathrm{T_N}(\omega'')$ assumes the existence of a non-singular $\mathrm{q}^2 \times \mathrm{q}^2$ numerical matrix $\mathrm{R}(\mathbf{w})$ which together with the transition operator fulfill the renowned Yang–Baxter relation [1],

$$\mathrm{R}(\mathbf{w})\left[\mathrm{L}\left(\omega'\right) \otimes \mathrm{I_q}\right]\left[\mathrm{I_q} \otimes \mathrm{L}\left(\omega''\right)\right] = \left[\mathrm{I_q} \otimes \mathrm{L}\left(\omega''\right)\right]\left[\mathrm{L}\left(\omega'\right) \otimes \mathrm{I_q}\right]\mathrm{R}(\mathbf{w}), \tag{4}$$

where $\mathrm{I_q}$ denotes the $\mathrm{q} \times \mathrm{q}$ unity matrix and the tensor product is considered within the auxiliary space. We have used the bold symbol $\mathbf{w}$ to emphasize that the entries of the R-matrix should not be confused with the set of Boltzmann weights ω defining the transition operator.