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Integral invariants in flat superspace ☆

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Abstract

We are solving for the case of flat superspace some homological problems that were formulated by Berkovits and Howe. (Our considerations can be applied also to the case of supertorus.) These problems arise in the attempt to construct integrals invariant with respect to supersymmetry. They appear also in other situations, in particular, in the pure spinor formalism in supergravity.

1. Introduction

In present paper we are solving for the case of flat superspace some homological problems that were formulated in the paper [1]. (Our considerations can be applied also to the case of supertorus.) These problems arise in the attempt to construct integrals invariant with respect to supersymmetry and in other situations.

Let us consider a flat superspace with coordinates $z^M \sim (x^m, \theta^\alpha)$. The supersymmetry Lie algebra *susy* is generated by transformations

$$[e_\alpha,\ e_\beta]_+ = \gamma^m_{\alpha\beta} P_m, \quad \text{with differential } d = \gamma^m_{\alpha\beta} t^\alpha t^\beta \frac{\partial}{\partial C^m}$$

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Here e_α are generators acting on the space of (x^m, θ^α), P_m is the translation operator, and $\gamma^m_{\alpha\beta}$ are Dirac Gamma matrices.

To construct integrals invariant with respect to supersymmetry one should find closed differential forms expressed in terms of physical fields. This is the homological problem we are trying to solve (see [1] for more details).

We consider a basis $E^m = E^m_M dz^M$, $E^\alpha = E^\alpha_M dz^M$ in the space of one-forms given by formulas $E^\alpha = d\theta^\alpha$, $E^m = dx^m + \Gamma^m_{\alpha\beta}\theta^\alpha d\theta^\beta$. Every differential form has a unique decomposition $\omega = \sum \omega_{p,q}$, where

$$\omega_{p,q} = \frac{1}{p!q!} E^{\beta_q} \cdots E^{\beta_1} E^{a_p} \cdots E^{a_1} \omega_{a_1 \cdots a_p \beta_1 \cdots \beta_q}(x, \theta)$$

In other words, the space Ω of all forms is a direct sum of subspaces $\Omega_{p,q}$.

We can consider differential forms as functions depending on even variables x^m, E^α and odd variables θ^α, E^m. Using the relations $dE^\alpha = 0$, $dE^m = \Gamma^m_{\alpha\beta} E^\alpha E^\beta$ we obtain a representation of the exterior differential d in the form

$$d = d_0 + d_1 + t_0 + t_1$$

with bi-degrees: $d_0 \sim (1, 0)$, $d_1 \sim (0, 1)$, $t_0 \sim (-1, 2)$, $t_1 \sim (2, -1)$.

It follows from $d^2 = 0$ that

$$t_0^2 = 0 \tag{1}$$

$$d_1 t_0 + t_0 d_1 = 0 \tag{2}$$

$$d_1^2 + d_0 t_0 + t_0 d_0 = 0 \tag{3}$$

This means, in particular, that t_0 can be considered as a differential; corresponding cohomology groups will be denoted by $H_t^{p,q}$. The differential t_0 can be identified with the differential

$$\gamma^m_{\alpha\beta} t^\alpha t^\beta \frac{\partial}{\partial C^m}$$

appearing in the calculation of cohomology group of the supersymmetry algebra *susy*; hence the cohomology groups $H_t^{p,q}$ coincide with graded components of cohomology groups of the Lie superalgebra *susy*.

Cohomology groups of the Lie superalgebra *susy* were calculated in [7], this allows us to compute the groups $H_t^{p,q}$. However, this is not the end of the story: the operator d_1 induces a differential on $H_t^{p,q}$; corresponding homology groups are denoted by $H_s^{p,q}$. (See [1].) We would like to calculate these groups. We give complete answers for zero momentum (independent of coordinate variables x^m) part of $H_s^{p,q}$. (It will be denoted by $\mathcal{H}_s^{p,q}$.)

Ordinary supersymmetry superspace, which we used in the above construction, can be replaces by a superspace that supports extended supersymmetry. We analyze the case of $N = 2$ supersymmetry in ten-dimensional space. Notice that in this case the group $\mathcal{H}_s^{0,q}$ can be interpreted as the cohomology of the differential $t_L \frac{\partial}{\partial\theta_L} + t_R \frac{\partial}{\partial\theta_R}$ where the spinors t_L, t_R obey the relaxed pure spinor condition $t_L \gamma^m t_L + t_R \gamma^m t_R = 0$. N. Berkovits informed us that the problem of calculation of this cohomology arises in pure spinor formalism of ten-dimensional supergravity; Section 3.2 answers his question. The same cohomology appears also in [8]

Notice, that in particular cases the groups $H_s^{p,q}$ appeared earlier under the name of pure spinor and spinorial cohomology, see Refs. [2–5]. For the discussion of the relation between $H_s^{p,q}$, pure spinor and spinorial cohomology see Ref. [6].