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Nuclear Physics B 884 (2014) 74–105

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Holographic Brownian motion in 1 + 1 dimensions

Pinaki Banerjee *, B. Sathiapalan

Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai-600113, India

Received 5 December 2013; received in revised form 13 April 2014; accepted 17 April 2014

Available online 24 April 2014

Editor: Hubert Saleur

Abstract

We study the motion of a stochastic string in the background of a BTZ black hole. In the $1+1$ dimensional boundary theory this corresponds to a very heavy external particle (e.g., a quark), interacting with the fields of a CFT at finite temperature, and describing Brownian motion. The equations of motion for a string in the BTZ background can be solved exactly. Thus we can use holographic techniques to obtain the Schwinger–Keldysh Green function for the boundary theory for the force acting on the quark. We write down the generalized Langevin equation describing the motion of the external particle and calculate the drag and the thermal mass shift. Interestingly we obtain dissipation even at zero temperature for this $1+1$ system. Even so, this does not violate boost (Lorentz) invariance because the drag force on a *constant* velocity quark continues to be zero. Furthermore since the Green function is exact, it is possible to write down an effective membrane action, and thus a Langevin equation, located at a "stretched horizon" at an arbitrary finite distance from the horizon.

1. Introduction

AdS/CFT correspondence [1–3,29] has been used quite successfully to study thermal properties such as the viscosity of $\mathcal{N}=4$ super Yang–Mills theory at finite temperature. Dissipation and thermal fluctuation are two sides of the same coin as embodied in the famous fluctuation–dissipation (FD) theorem. The study of fluctuations using holographic techniques has been done

* Corresponding author.
E-mail addresses: pinakib@imsc.res.in (P. Banerjee), bala@imsc.res.in (B. Sathiapalan).

http://dx.doi.org/10.1016/j.nuclphysb.2014.04.016

in several papers [4–7,9–11,13] and the fluctuation–dissipation theorem has been shown to hold. Different techniques [4,5] have been used to address this issue. A very versatile technique is in terms of Green functions. Son and Teaney [5] have used holographic techniques to calculate Green functions to address these questions in the context of Brownian motion of a particle such as a quark.

The fluctuation–dissipation theorem in the context of Brownian motion has been studied by Kubo [17,18] and Mori [19,20] among others. Brownian motion can be described as a stochastic process [21]. In some approximation it is Markovian. If we can assume that velocities at two instants are not correlated, then it is a Markovian process when described in terms of position. Thus one can define a probability $P(x(t),t;x(t_0),t_0)$ as the conditional probability for the particle to be in position $x(t)$ at time t given that it was at $x(t_0)$ at time t_0. One can also write a Fokker–Planck equation for $P(x(t),t;x(t_0),t_0)$. On the other hand if we want a finer description one can use the velocity as the variable defining the Markovian process in terms of $P(v(t),t;v(t_0),t_0)$. This is a good approximation as long as the duration of a collision is very small, which is equivalent to saying that acceleration at different instants is uncorrelated. The Fokker–Planck equation in the velocity description is

$$\frac{\partial P(v,t)}{\partial t} = -\frac{\partial}{\partial v}a_1(v)P + \frac{a_2}{2}\frac{\partial^2 P}{\partial v^2} \tag{1.1}$$

Here $a_1 = \frac{\langle \Delta v\rangle}{\Delta t}$ and $a_2 = \frac{\langle (\Delta v)^2\rangle}{\Delta t}$. Here Δv is the change in velocity in time Δt.

One can obtain these from the related Langevin equation:

$$m\dot{v} = -\gamma v + \xi(t) \tag{1.2}$$

where $\xi(t)$ is the random force that is responsible for the fluctuations, obeying $\langle \xi(t)\xi(t')\rangle = \Gamma\delta(t-t')$ and $\langle \xi(t)\rangle = 0$. $v(t_0) = v_0$ is the initial condition. Thus $a_1 = \langle v(\Delta t) - v_0\rangle = -\frac{\gamma}{m}v_0\Delta t$. From the solution of the Langevin equation (taking $t_0 = 0$):

$$v(t) = v_0 e^{-\frac{\gamma}{m}t} + \frac{1}{m}\int_0^t e^{-\frac{\gamma}{m}(t-t')}\xi\left(t'\right)dt' \tag{1.3}$$

one can obtain $a_2 = \frac{\Gamma}{m^2}$. Thus the Fokker–Planck equation becomes:

$$\frac{\partial P(v,t)}{\partial t} = \frac{\gamma}{m}\frac{\partial}{\partial v}vP + \frac{\Gamma}{2m^2}\frac{\partial^2 P}{\partial v^2} \tag{1.4}$$

Finally since we know that $P(v) = e^{-\frac{mv^2}{2kT}}$ is a time independent solution of the Fokker–Planck equation we get

$$\Gamma = 2\gamma kT \tag{1.5}$$

This is the fluctuation–dissipation theorem in this context, because it relates Γ, the strength of the fluctuation, to γ the strength of the dissipation.

The Langevin equation is much more convenient to work with. To the extent that it assumes that time scales are larger than the microscopic time scale it must fail for very small time scales. As Kubo has shown, stationarity should imply that

$$\frac{d}{dt_0}\left\langle v(t_0)v(t_0)\right\rangle = 0 = \left\langle \dot{v}(t_0)v(t_0)\right\rangle$$