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Notes on Mayer expansions and matrix models

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Abstract

Mayer cluster expansion is an important tool in statistical physics to evaluate grand canonical partition functions. It has recently been applied to the Nekrasov instanton partition function of $\mathcal{N}=2$ 4d gauge theories. The associated canonical model involves coupled integrations that take the form of a generalized matrix model. It can be studied with the standard techniques of matrix models, in particular collective field theory and loop equations. In the first part of these notes, we explain how the results of collective field theory can be derived from the cluster expansion. The equalities between free energies at first orders is explained by the discrete Laplace transform relating canonical and grand canonical models. In a second part, we study the canonical loop equations and associate them with similar relations on the grand canonical side. It leads to relate the multi-point densities, fundamental objects of the matrix model, to the generating functions of multi-rooted clusters. Finally, a method is proposed to derive loop equations directly on the grand canonical model.

1. Introduction

The AGT correspondence [1] implies a relation between the canonical partition function of a β-ensemble and the grand canonical partition function of a generalized matrix model. The former represents a correlator of Liouville theory, according to the proposal of Dijkgraaf and Vafa [2], further investigated in [3–11]. The latter describes the instanton partition function of a 4d $\mathcal{N}=2$ supersymmetric gauge theory in the Ω-background, as derived using localization techniques in [12]. Here the term 'generalized matrix model' do not pertain to a matrix origin

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for the model, but instead refers to a set of models that can be studied using techniques initially developed in the realm of matrix models. Among these techniques, the *topological recursion* [13] exploits the invariance of the integration measure to derive a tower of nested equations satisfied by the correlators of the model. These equations, referred as *loop equations*, are solved employing methods from algebraic geometry. This technique has recently been extended to a wide spectrum of coupled integrals models in [14].

In a suitable limit of the β-ensemble, AGT-equivalent to the Nekrasov–Shatashvili (NS) limit of the Ω-background [15], loop equations are no longer algebraic but first order linear differential equations.[1] In this context, the β-ensemble is a natural quantization of the Hermitian matrix model, to which it reduces at $\beta = 1$. The first element of this tower of differential equations has been mapped to the TQ relation derived in [16–18] that describes the dual SUSY gauge theory in the NS limit [19–22]. It is then natural to ask about the existence of a structure similar to loop equations on the gauge side of the correspondence.[2] But so far, the loop equation technique has not been applied to grand canonical matrix models. On the other hand, the cluster expansion of Mayer and Montroll [26] has been successfully employed to derived an effective action relevant to the NS limit [15]. Can we relate this cluster expansion to the topological expansion of a generalized matrix model? Is there an equivalent of the loop equations technique on the grand canonical side? And more generally, how do canonical and grand canonical coupled integrals relate to each other? These are the issues we propose to address in these notes.

For this purpose, we consider the following grand canonical generalized matrix model,

$$\mathcal{Z}_{\mathrm{GC}}(\bar{q}) = \sum_{N=0}^{\infty} \frac{\bar{q}^N}{N!} \mathcal{Z}_{\mathrm{C}}(N), \quad \mathcal{Z}_{\mathrm{C}}(N) = \int_{\mathbb{R}^N} \prod_{i=1}^{N} Q(\phi_i) \frac{d\phi_i}{2i\pi} \prod_{\substack{i,j=1\\ i<j}}^{N} K(\phi_i - \phi_j). \tag{1.1}$$

In analogy with the Nekrasov partition function, integrals are understood as contour integrals over the real line. The potential $Q(x)$ and the kernel $K(x)$ are free of singularities over the real axis.[3] We propose to study the expansion of $\mathcal{Z}_{\mathrm{GC}}(q)$ when the kernel is close to one. More precisely, we assume the form

$$K(x) = 1 + \epsilon f(x), \quad \epsilon \to 0, \tag{1.2}$$

with f an even function, non-vanishing at $x = 0$. Although the results of these notes are very general, what we have in mind for the function f is typically

$$f(x) = \frac{1}{x^2 - \gamma^2}, \quad \mathrm{Im}\,\gamma \neq 0. \tag{1.3}$$

It is crucial for our considerations that f is independent of ϵ. In this way, we exclude a class of models more relevant to the study of Nekrasov partition functions. For instance, setting $\epsilon = \gamma^2$, one recovers the model proposed by J. Hoppe in [27]. This model is a one-parameter version of the Nekrasov partition function that depends on two Ω-background equivariant deformation parameters ϵ_1 and ϵ_2 [28,29]. As $\epsilon \to 0$, it exhibits a phenomenon referred as *instanton clustering*

[1] Except for the first (planar) equation, which is a Riccati equation, therefore non-linear. It is equivalent to a Schrödinger equation, i.e. a linear differential equation of second order.

[2] Such a structure should be related to the invariance of Nekrasov partition functions under transformations representing the SHc algebra uncovered in [23] (see also [24,25]).

[3] In the case of real singularities, a prescription should be given to move away the poles from the contour by a small imaginary shift.