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Scattering theory of the hyperbolic BC_n Sutherland and the rational BC_n Ruijsenaars–Schneider–van Diejen models

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Abstract

In this paper, we investigate the scattering properties of the hyperbolic BC_n Sutherland and the rational BC_n Ruijsenaars–Schneider–van Diejen many-particle systems with three independent coupling constants. Utilizing the recently established action-angle duality between these classical integrable models, we construct their wave and scattering maps. In particular, we prove that for both particle systems the scattering map has a factorized form.

Keywords: Scattering theory; Integrable systems; Pure soliton systems

1. Introduction

The Calogero–Moser–Sutherland (CMS) many-particle systems (see e.g. [1–4]) and their relativistic deformations, the Ruijsenaars–Schneider–van Diejen (RSvD) models (see e.g. [5,6]) are among the most actively studied integrable systems. They appear in several branches of mathematics and physics, with numerous applications ranging from symplectic geometry, Lie theory and harmonic analysis to solid state physics and Yang–Mills theory. The intimate connection with the theory of soliton equations is a particularly important and appealing feature of these finite dimensional integrable systems. It is a remarkable fact that the CMS and the RSvD models associated with the A_n root system can be used to describe the soliton interactions of certain integrable field theories defined on the whole real line (see e.g. [5,7–10]). In particular, these particle

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systems are characterized by conserved asymptotic momenta and factorized scattering maps (see e.g. [7,11–13]). That is, in perfect analogy with the behavior of the solitons, the n-particle scattering is completely determined by the 2-particle processes. However, apart form some heuristic arguments [14], the link between the integrable boundary field theories and the particle models associated with the non-A_n-type root systems has not been developed yet. In our paper [15] we have proved that the classical hyperbolic C_n Sutherland model also has a factorized scattering map, but to our knowledge analogous results for the other non-A_n-type root systems are not available. Motivated by this fact, in this paper we work out in detail the scattering theory of the hyperbolic BC_n Sutherland and the rational BC_n RSvD models with the maximal number of independent coupling constants.

Upon introducing the subset

$$\mathfrak{c} = \big\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \,\big|\, x_1 > \cdots > x_n > 0\big\} \subset \mathbb{R}^n, \tag{1.1}$$

let us recall that the classical hyperbolic BC_n Sutherland and the rational BC_n RSvD models live on the phase spaces

$$\mathcal{P}^S = \big\{(q, p) \,\big|\, q \in \mathfrak{c},\ p \in \mathbb{R}^n\big\} \quad \text{and} \quad \mathcal{P}^R = \big\{(\lambda, \theta) \,\big|\, \lambda \in \mathfrak{c},\ \theta \in \mathbb{R}^n\big\}, \tag{1.2}$$

respectively. By endowing these spaces with the natural symplectic forms

$$\omega^S = \sum_{a=1}^n \mathrm{d}q_a \wedge \mathrm{d}p_a \quad \text{and} \quad \omega^R = \sum_{a=1}^n \mathrm{d}\lambda_a \wedge \mathrm{d}\theta_a, \tag{1.3}$$

we may think of $\mathcal{P}^S$ and $\mathcal{P}^R$ as two different copies of the cotangent bundle $T^*\mathfrak{c}$. The hyperbolic BC_n Sutherland model is characterized by the interacting many-body Hamiltonian

$$H^S = \sum_{a=1}^n \left(\frac{p_a^2}{2} + g_1^2 w(q_a) + g_2^2 w(2q_a)\right) + \sum_{1 \leqslant a < b \leqslant n} \big(g^2 w(q_a - q_b) + g^2 w(q_a + q_b)\big) \tag{1.4}$$

with potential function $w(x) = \sinh(x)^{-2}$. To ensure the purely repulsive nature of the interaction, on the real parameters g, g_1 and g_2 we impose $g^2 > 0$ and $g_1^2 + g_2^2 > 0$. As for the rational BC_n RSvD model, the dynamics is governed by the Hamiltonian

$$H^R = \sum_{a=1}^n \cosh(2\theta_a) v_a(\lambda) + \frac{\nu\kappa}{4\mu^2} \prod_{a=1}^n \left(1 + \frac{4\mu^2}{\lambda_a^2}\right) - \frac{\nu\kappa}{4\mu^2}, \tag{1.5}$$

where the real parameters μ, ν and κ appear in the function

$$v_a(\lambda) = \left(1 + \frac{\nu^2}{\lambda_a^2}\right)^{\frac{1}{2}} \left(1 + \frac{\kappa^2}{\lambda_a^2}\right)^{\frac{1}{2}} \prod_{\substack{b=1 \\ (b \neq a)}}^n \left(1 + \frac{4\mu^2}{(\lambda_a - \lambda_b)^2}\right)^{\frac{1}{2}} \left(1 + \frac{4\mu^2}{(\lambda_a + \lambda_b)^2}\right)^{\frac{1}{2}} \tag{1.6}$$

as well. Let us note that on the RSvD coupling parameters we impose the constraints $\mu \neq 0$, $\nu \neq 0$ and $\nu\kappa \geqslant 0$.

Working in a symplectic reduction framework, in our paper [16] we established the action-angle duality between the hyperbolic BC_n Sutherland and the rational BC_n RSvD models, provided the coupling parameters satisfy the relations