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Off-diagonal Bethe ansatz solution of the XXX spin chain with arbitrary boundary conditions

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Abstract

Employing the off-diagonal Bethe ansatz method proposed recently by the present authors, we exactly diagonalize the XXX spin chain with arbitrary boundary fields. By constructing a functional relation between the eigenvalues of the transfer matrix and the quantum determinant, the associated $T-Q$ relation and the Bethe ansatz equations are derived.

Keywords: Spin chain; Reflection equation; Bethe ansatz; $T-Q$ relation

1. Introduction

Our understanding of quantum phase transitions and critical phenomena has been greatly enhanced by the study of exactly solvable models (or quantum integrable systems) [1]. Such exact results have provided valuable insight into the important universality classes of quantum physical systems ranging from modern condensed matter physics [2] to string and super-symmetric Yang–Mills theories [3]. Since Yang and Baxter's pioneering works [4,5,1], the quantum Yang–Baxter equation (QYBE), which defines the underlying algebraic structure, has become a cornerstone for constructing and solving the integrable models. There are several well-known methods for deriving the Bethe ansatz (BA) solution of integrable models: the coordinate BA [6,1,7–9], the

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T–Q approach [1,10], the algebraic BA [11–13], the analytic BA [14], the functional BA [15] and others [16–29].

Generally speaking, there are two classes of integrable models. One possesses $U(1)$ symmetry and the other does not. Three well-known examples without $U(1)$ symmetry are the XYZ spin chain [5,12], the XXZ spin chain with antiperiodic boundary condition [30,31,25–27,29] and the ones with unparallel boundary fields [18–21,24–29]. It has been proven that most of the conventional Bethe ansatz methods can successfully diagonalize the integrable models with $U(1)$ symmetry. However, for those without $U(1)$ symmetry, only some very special cases such as the XYZ spin chain with even site number [5,12] and the XXZ spin chain with constrained unparallel boundary fields [19,20,32,33] can be dealt with due to the existence of a proper "local vacuum state" in these special cases. The main obstacle applying the algebraic Bethe ansatz and Baxter's method to general integrable models without $U(1)$ symmetry lies in the absence of such a "local vacuum". A promising method for approaching such kind of problems is Sklyanin's separation of variables method [15] which has been recently applied to some integrable models [26–29]. However, before the very recent work [34], a systematic method was still absent to derive the Bethe ansatz equations for integrable models without $U(1)$ symmetry, which are crucial for studying the physical properties in the thermodynamic limit.

As for integrable models without $U(1)$ symmetry, some off-diagonal elements of monodromy matrix enter into expression of the transfer matrix. This breaks down the usual $U(1)$ symmetry. Very recently, we have proposed a method [34] for dealing with the integrable models without $U(1)$ symmetry. The central idea of the method is to construct the functional relations between eigenvalues $\Lambda(\lambda)$ of the transfer matrix (the trace of the monodromy matrix) and its quantum determinant $\Delta_q(\lambda)$, i.e., $\Lambda(\theta_j)\Lambda(\theta_j-\eta)\sim\Delta_q(\theta_j)$ (see below (4.17)) based on the zero points of the product of off-diagonal elements of monodromy matrix $B(u)B(u-\eta)=0$. Since the trace and the determinant are two basic quantities of a matrix which are independent of the representation basis, this method could overcome the obstacle of absence of a reference state which is crucial in the conventional Bethe ansatz methods.

Our primary motivation for this work comes from the long standing problem of solving the open spin-$\frac{1}{2}$ XXX spin chain with unparallel boundary fields, defined by the Hamiltonian [35,36]

$$H=\sum_{j=1}^{N-1}\vec{\sigma}_j\vec{\sigma}_{j+1}+h_N\sigma_N^z+h_1^x\sigma_1^x+h_1^z\sigma_1^z. \tag{1.1}$$

N is the site number of the system and σ_j^α ($\alpha=x,y,z$) is the Pauli matrix on the site j along the α direction. The parameters h_N, h_1^x and h_1^z are related to boundary fields. Solving this problem for generic values of these three parameters is a crucial step in formulating the thermodynamics of the spin chain, due to the fact that this problem has important applications in condensed matter physics and statistical mechanics. In this paper, we shall use the method developed in [34] to solve the eigenvalue problem of the above Hamiltonian with generic h_N, h_1^x and h_1^z.

The paper is organized as follows. Section 2 serves as an introduction of our notation and some basic ingredients. We briefly describe the inhomogeneous open XXX chain with non-diagonal boundary terms. In Section 3, we derive the exchange relations among the matrix entries of the monodromy matrix algebras (or the Yang–Baxter algebras). In Section 4 after obtaining some properties of the eigenvalue as a function of spectrum parameter u, we derive the very relation between the eigenvalue and the quantum determinant of the double-row monodromy matrix. This allows us to construct a generalized T–Q relation type solution of eigenvalue. In Section 5, we consider the homogeneous limit of the results of the previous section and give the energy