Available online at www.sciencedirect.com
ScienceDirect
Nuclear Physics B 852 [FS] (2011) 269–286
NUCLEAR PHYSICS B
www.elsevier.com/locate/nuclphysb

On the normalization of the partition function of Bethe Ansatz systems

F. Woynarovich

Institute for Solid State Physics and Optics, Hungarian Academy of Sciences, 1525 Budapest 114, Pf 49, Hungary

Received 17 June 2011; accepted 24 June 2011

Abstract

In this note I revisit the calculation of partition function of simple one-dimensional systems solvable by Bethe Ansatz. Particularly I show that by the precise definition and treatment of the partition function the nontrivial normalization factor proposed in a recent work to give the correct $O(1)$ corrections to the free energy can be derived in a straightforward manner.

Keywords: Integrable systems; Thermodynamic Bethe Ansatz; Finite size correction

1. Introduction

Recent developments in both field theory and solid state physics have shown, that in certain problems, where surface or impurity effects are important, non-macroscopic contributions to the thermodynamic quantities like the free energy may play an important role [1,2]. Among these systems those solvable by Bethe Ansatz (BA) due to the exact treatability are of special importance. For these one-dimensional (1D) systems the free energy is calculated following the method developed by C.N. Yang and C.P. Yang [3] for the δ Bose gas. The basic idea of this method is that through the density of momenta (rapidities) an entropy can be defined and a free energy as a functional of the rapidity density can be constructed. The minimization of this functional yields both the equilibrium state and the macroscopic part of the free energy of the finite temperature system. Based on this idea the present author developed a method to calculate the $O(1)$ free energy contributions of the states near to the equilibrium [4], however the contributions found have

---

E-mail address: fw@szfki.hu.

doi:10.1016/j.nuclphysb.2011.06.015

not met the expectations [5]: for periodic boundary conditions (PBC) no $O(1)$ corrections have been expected, but the calculation gave some, and also for the case of open ends with integrable boundaries (IB) some of the obtained terms were not of the expected structure. Recently, based on intuitive arguments concerning the density of states in the configuration space a nontrivial normalization of the partition function has been proposed [6], by which these differences can be dissolved: in the case of PBC the corrections are canceled, while for the IB case they are corrected. In the present note we derive this nontrivial factor directly by the careful definition and treatment of the partition function. This way this work confirms the proposal of [6] and completes [4].

As a starting point we briefly review the ingredients of the calculations. This will serve also to make clear our notations and expose the problem in a more tractable form. It has been tempting to formulate our treatment in a general form, as however the derivation of the entropy term for the free energy functional is slightly different for closed and open boundaries we treat the two cases separately: first we deal with the case of periodic boundary condition in more details, and in case of integrable boundaries we point out the differences only.

2. BA and BA thermodynamics

We consider a system with BA equations

$$Lp(\theta_i)+\sum_{j=1}^{N}\phi(\theta_i-\theta_j)=2\pi I_i. \tag{1}$$

Here the θ are the rapidities of the particles, $p(\theta)$ is the momentum of a particle with rapidity θ, the $\phi(\theta-\theta')$ is a phase closely related (up to a constant equal) to the phase shift arising when a particle with rapidity θ is scattered on an other with θ', and the I quantum numbers are either integers or halfs of odd-integers depending on the number of particles N. (For later purposes we chose the Riemann sheets to have ϕ continuous at zero argument.) To each set of quantum numbers $\{I_j, I_j\neq I_l,\ \text{if } j\neq l\}$ (1) defines a set of real rapidities $\{\theta_j,\theta_j\neq\theta_l,\ \text{if } j\neq l\}$. The wave functions belonging to the different solutions of (1) are orthogonal and form a complete set. The energy of such a state (modified due to the chemical potential μ if necessary) is the sum of the contributions of the individual particles

$$E-\mu N=E(\theta_1,\theta_2,\dots,\theta_N)=\sum_{i=1}^{N}e(\theta_i). \tag{2}$$

The finite temperature description [3] is based on the idea, that for a macroscopic system the roots of (1) can be described by densities, and the thermodynamic quantities can be given by these. In particular the $\rho(\theta)$ density of particles is defined so, that the # of $\theta_i\in(\theta,\theta+\Delta\theta)$ is $L\rho(\theta)\Delta\theta$ and the $\rho_h(\theta)$ density of holes is given by the equation obtained from (1)

$$\rho(\theta)+\rho_h(\theta)=\frac{1}{2\pi}\frac{\partial p(\theta)}{\partial\theta}+\int K\bigl(\theta-\theta'\bigr)\rho\bigl(\theta'\bigr)\,d\theta'\quad\text{with } K(\theta)=\frac{1}{2\pi}\frac{\partial\phi(\theta)}{\partial\theta}. \tag{3}$$

All $\rho(\theta)$ functions are physical, for which (3) yields a non-negative $\rho_h(\theta)$, as for these densities I_i quantum number sets can be constructed which define θ_i roots distributed according to $\rho(\theta)$ (with a certain accuracy). The number how many ways this can be done, i.e. the number of states represented by one single rapidity density is estimated by the combinatorial factor