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Graph fusion algebras of $\mathcal{WLM}(p, p')$

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Abstract

We consider the $\mathcal{W}$-extended logarithmic minimal model $\mathcal{WLM}(p, p')$. As in the rational minimal models, the so-called fundamental fusion algebra of $\mathcal{WLM}(p, p')$ is described by a simple graph fusion algebra. The fusion matrices in the regular representation thereof are mutually commuting, but in general not diagonalizable. Nevertheless, we show that they can be brought simultaneously to block-diagonal forms whose blocks are upper-triangular matrices of dimension 1, 3, 5 or 9. The directed graphs associated with the two fundamental modules are described in detail. The corresponding adjacency matrices share a complete set of common generalized eigenvectors organized as a web constructed by interlacing the Jordan chains of the two matrices. This web is here called a Jordan web and it consists of connected subwebs with 1, 3, 5 or 9 generalized eigenvectors. The similarity matrix, formed by concatenating these vectors, simultaneously brings the two fundamental adjacency matrices to Jordan canonical form modulo permutation similarity. The ranks of the participating Jordan blocks are 1 or 3, and the corresponding eigenvalues are given by $2\cos\frac{j\pi}{\rho}$ where $j = 0, \ldots, \rho$ and $\rho = p, p'$. For $p > 1$, only some of the modules in the fundamental fusion algebra of $\mathcal{WLM}(p, p')$ are associated with boundary conditions within our lattice approach. The regular representation of the corresponding fusion subalgebra has features similar to the ones in the regular representation of the fundamental fusion algebra, but with dimensions of the upper-triangular blocks and connected Jordan-web components given by 1, 2, 3 or 8. Some of the key results are illustrated for $\mathcal{W}$-extended critical percolation $\mathcal{WLM}(2, 3)$.

1. Introduction

A central question of much current interest is whether an extended symmetry algebra $\mathcal{W}$ [1,2] exists for logarithmic conformal field theories [3–6] like the logarithmic minimal models

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$\mathcal{LM}(p, p')$ [7–9]. These models contain a countably *infinite* number of inequivalent Virasoro modules which the extended symmetry should reorganize into a *finite* number of $\mathcal{W}$-extended modules closing under fusion. In the case of the logarithmic minimal models $\mathcal{LM}(1, p')$, the existence and properties of such an extended $\mathcal{W}$-symmetry, including the associated fusion rules, are by now largely understood [10–17]. The works [18,19] strongly indicate the existence of a $\mathcal{W}_{p,p'}$ symmetry algebra for general augmented minimal models, but offer only very limited insight into the associated fusion algebras. Recently, a detailed description of these fusion algebras has been provided in [20–22] generalizing the approach of [17]. Extending ideas originating with Cardy [23,24], this approach uses a strip-lattice implementation of fusion to obtain the fusion rules of the entire series of logarithmic minimal models $\mathcal{LM}(p, p')$ in the $\mathcal{W}$-extended picture where they are denoted by $\mathcal{WLM}(p, p')$. It is stressed, that the extended picture is described by the *same* lattice model as the Virasoro picture.

Contrary to the situation in the Virasoro picture, for $p > 1$, there is no identity nor a pair of so-called fundamental modules in the lattice approach to $\mathcal{WLM}(p, p')$. In [22], we found that one can supplement the set of indecomposable modules associated with boundary conditions by a set of reducible yet indecomposable rank-1 modules. This algebraically enlarged set was shown to yield a well-defined fusion algebra called the *fundamental fusion algebra*. This algebra is so named since it is generated from repeated fusions of the two *fundamental modules* $(2, 1)_{\mathcal{W}}$ and $(1, 2)_{\mathcal{W}}$ in addition to the identity $(1, 1)_{\mathcal{W}}$ which is now present for all p. It was also found that the fusion algebra generated by modules associated with boundary conditions is an *ideal* of the fundamental fusion algebra. Further algebraic extensions exist. In particular, for $p > 1$, there are additional *irreducible* modules not associated with boundary conditions. Their fusion properties have been systematically examined only very recently [25–27]. Here we restrict ourselves to the modules generating the fundamental fusion algebra.

The fusion matrices of a standard rational conformal field theory are diagonalizable. This is made manifest by the Verlinde formula [28] where the diagonalizing similarity matrix is the modular S-matrix of the characters in the theory. In a logarithmic conformal field theory, on the other hand, there are typically more linearly independent representations than linearly independent characters due to the presence of indecomposable modules of rank greater than 1. Consequently, there is no Verlinde formula in the usual sense and the fusion matrices may not all be diagonalizable. This is indeed the situation for the $\mathcal{W}$-extended logarithmic minimal models $\mathcal{WLM}(p, p')$ analyzed in the present work.

In the regular representation of a fusion algebra, the fusion matrices are mutually commuting. Viewing the fusion matrices as adjacency matrices of graphs, the fusion rules are succinctly encoded in these fusion graphs. In this context, the regular representation of a fusion algebra is referred to as the *graph fusion algebra*. Fusion graphs have been the key to the classification of rational conformal field theories on the cylinder [29,30] and on the torus [31–34]. In the rational A-type theories, the Verlinde algebra yields a diagonal modular invariant, while D- and E-type theories are related to non-diagonal modular invariants. The Ocneanu algebras arise when considering fusion on the torus, with left and right chiral halves of the theory, and involve Ocneanu graphs. We refer to [35–38] for earlier results on the interrelation between fusion algebras, graphs and modular invariants. It is our hope that the present work will be a step in the direction of extending these fundamental insights to the logarithmic conformal field theories.

As already indicated, the fusion matrices in the regular representation of the fundamental fusion algebra are mutually commuting, but in general not diagonalizable. Nevertheless, we show that they can be brought simultaneously to block-diagonal forms whose blocks are upper-triangular matrices of dimension 1, 3, 5 or 9. The directed graphs associated with the two