Available online at www.sciencedirect.com
ScienceDirect
Nuclear Physics B 829 [FS] (2010) 573–599
NUCLEAR PHYSICS B
www.elsevier.com/locate/nuclphysb

Critical loop gases and the worm algorithm

Wolfhard Janke [a], Thomas Neuhaus [b], Adriaan M.J. Schakel [a,*]

[a] *Institut für Theoretische Physik, Universität Leipzig, Postfach 100 920, D-04009 Leipzig, Germany*
[b] *Jülich Supercomputing Centre, Forschungszentrum Jülich, D-52425 Jülich, Germany*

Received 27 October 2009; accepted 22 December 2009

Available online 5 January 2010

Abstract

The loop gas approach to lattice field theory provides an alternative, geometrical description in terms of fluctuating loops. Statistical ensembles of random loops can be efficiently generated by Monte Carlo simulations using the worm update algorithm. In this paper, concepts from percolation theory and the theory of self-avoiding random walks are used to describe estimators of physical observables that utilize the nature of the worm algorithm. The fractal structure of the random loops as well as their scaling properties are studied. To support this approach, the O(1) loop model, or high-temperature series expansion of the Ising model, is simulated on a honeycomb lattice, with its known exact results providing valuable benchmarks.

Keywords: Loop gas; Monte Carlo; Worm update algorithm; Fractal structure; Critical properties; Duality

1. Introduction

Representing the hopping of particles from one lattice site to the next, the strong-coupling expansion in relativistic quantum field theories formulated on a spacetime lattice provides an alternative approach to numerically simulating lattice field theories in terms of world lines. The standard approach, which is rooted in the functional integral approach to field quantization, involves estimating observables (expressed in terms of the fields) by sampling a representative set of field configurations. New configurations are typically generated by means of a Monte Carlo technique which uses importance sampling, with each field configuration weighted according to the probability that it occurs. In contrast, the approach based on the strong-coupling, or hopping

* Corresponding author.
E-mail address: schakel@itp.uni-leipzig.de (A.M.J. Schakel).

0550-3213/$ – see front matter
doi:10.1016/j.nuclphysb.2009.12.024

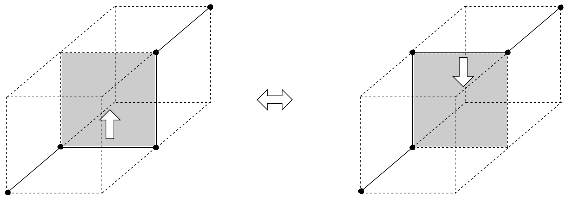

Fig. 1. Bond-shifting algorithm for generating a new world line configuration on a cubic lattice. Lattice sites visited by the walk are marked by full circles and the updated plaquettes are shaded.

expansion, which is closely connected to Feynman's spacetime approach to quantum theory [1], involves linelike objects. Physical observables are in this geometrical approach no longer estimated by sampling an ensemble of field configurations, but by sampling a grand canonical ensemble of (mostly closed) world lines, known as a *loop gas*, instead. The weight of a given world line configuration is typically determined by the total length of the paths, the number of intersections, and the number of loops contained in the tangle.

In statistical physics, the strong-coupling expansion is known as the high-temperature series expansion [2]. Lattice field theories studied in this context are typically spin models, such as the O(N) spin model, whose representation in terms of high-temperature (HT) graphs is known as a *loop model*.

A first numerical study of loop gases formulated on the lattice was carried out by Berg and Foerster [3]. New world line configurations were generated by a bond-shifting Monte Carlo update algorithm as follows. A randomly chosen bond of the existing configuration is shifted perpendicular to itself by one lattice spacing in any of the $2(d-1)$ directions of the hypercubic lattice. During the shift, each of the endpoints of the moving link erases or draws a bond in the chosen perpendicular direction, depending on whether the link is occupied or not, as in Fig. 1. The new configuration is accepted or rejected according to the Metropolis algorithm.

At about the same time, Dasgupta and Halperin [4], following a suggestion by Helfrich and Müller [5] that the HT graphs of the O(N) lattice model simultaneously describe a loop gas of sterically interacting physical lines, simulated a gas of directed loops on a cubic lattice. New loop configurations were generated in this study by inserting an elementary loop, or *plaquette*, of random orientation according to the Metropolis algorithm.

Although these and related early loop gas update algorithms [6–8] work fine in the disordered phase away from the critical point, they all, being based on local updates, suffer from pronounced critical slowing down. That is, consecutive configurations are highly correlated close to the critical point, and simulations on larger lattices become increasingly unfeasible in this region.

About a decade ago, Prokof'ev and Svistunov [9] have introduced a Monte Carlo update algorithm that, although based on local updates, does away with critical slowing down almost completely. The so-called *worm algorithm* generates loop configurations, not by inserting plaquettes, but through the motion of the end points of an *open* world line—the "head" and "tail" of a "worm". An additional loop is generated in this scheme when the head bites the tail, or through a "back bite" where the head erases a piece (bond) of its own body and thereby leaves behind a detached loop and a shortened open chain.

Besides this outstanding technical advantage, the worm algorithm has the additional advantage in the context of statistical physics that the complete set of standard critical exponents can be determined at a stroke. This set is known to split into two, *viz.* the thermal and the magnetic exponents. While the thermal exponents, such as the specific heat exponent α, pertain to closed