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The three-dimensional origin of the classifying algebra

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Abstract

It is known that reflection coefficients for bulk fields of a rational conformal field theory in the presence of an elementary boundary condition can be obtained as representation matrices of irreducible representations of the classifying algebra, a semisimple commutative associative complex algebra.

We show how this algebra arises naturally from the three-dimensional geometry of factorization of correlators of bulk fields on the disk. This allows us to derive explicit expressions for the structure constants of the classifying algebra as invariants of ribbon graphs in the three-manifold $S^2 \times S^1$. Our result unravels a precise relation between intertwiners of the action of the mapping class group on spaces of conformal blocks and boundary conditions in rational conformal field theories.

Keywords: Conformal field theory; Factorization; Boundary conditions; Classifying algebra; Conformal blocks

1. Introduction

Structure constants of operator product expansions (OPEs) have played an important role in shaping our understanding of correlation functions in two-dimensional conformal field theory. In fact, one approach to conformal field theory has been to identify a subset of fundamental correlators – which are ultimately encoded in appropriate OPEs – from which all other correlators can be obtained by sewing. Since a given correlation function can typically be constructed from the fundamental correlators by sewing in several distinct ways, the uniqueness of the correlators

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imposes various necessary conditions on the sewing procedure. These conditions are known as sewing constraints, or factorization constraints [1–3]. There are actually two different types of factorizations:

– *Boundary factorization*, involving a cutting of the world sheet along an interval that connects two points on its boundary, yields a correlator with two additional insertions of boundary fields.
– *Bulk factorization*, for which the cutting is along a circle in the interior of the world sheet, yields a correlator with two additional bulk field insertions.

In the guise of associativity of the OPE, constraints from bulk factorization have been central in the understanding of the OPE of bulk fields [4]. Much later it was realized [5,6] that the constraints on structure constants for boundary fields preserving a given boundary condition have in fact a simpler structure and give rise to an associative algebra in the tensor category of chiral data.

The OPE coefficients of bulk fields in the presence of a boundary are amenable as well. The one-point correlators of bulk fields on a disk, which may be collected in so-called boundary states, contain significant information of much interest for applications, like ground state degeneracies [7] or Ramond–Ramond charges of string compactifications [8]. Moreover, they provide essential information about annulus partition functions and thus encode the spectrum of boundary fields. Based on an analysis of specific classes of models, it was found that the reflection coefficients for bulk fields in the presence of an elementary boundary condition can be formulated in terms of representation matrices of irreducible representations of a *classifying algebra* [3,9].[1] In particular, the elementary boundary conditions are in bijection with the irreducible representations of the classifying algebra. Further, there is evidence that the structure constants of the classifying algebra can be expressed in terms of traces of intertwiners for the mapping class group action on spaces of conformal blocks on the sphere [11], whereby they are related to the subbundle structure of the bundles of conformal blocks [12]. These ideas have lead to concrete formulas for operator product coefficients [13–17] with a wide range of uses; see e.g. [18–21] for some applications in string theory.

More recently, the *TFT approach* to the correlation functions of rational conformal field theories has provided a much more satisfactory understanding of RCFT correlators. The main idea of this approach can be summarized as follows. The chiral data of a conformal field theory are described by the structure of a modular tensor category [22,23], and a full local conformal field theory based on these chiral data corresponds to (a Morita class of) a symmetric special Frobenius algebra in that category [24–28]. A modular tensor category also gives rise to a three-dimensional topological field theory [29]. The TFT approach uses this topological field theory to construct the correlators of the local conformal field theory as invariants of ribbon graphs in three-manifolds with boundary. It has been shown [30] that the so obtained correlators are invariant under the mapping class group and obey all factorization constraints.

The purpose of this paper is to establish the existence of a classifying algebra for any RCFT – an associative commutative algebra $\mathcal{A}$ over the complex numbers with the property that the homomorphisms given by its irreducible representations give the bulk reflection coefficients in the presence of elementary boundary conditions. We also show that this algebra

[1] The same algebra had arisen in the study of integrable lattice models [10].