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Critical point of the two-dimensional Bose gas: An S-matrix approach

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Abstract

A new treatment of the critical point of the two-dimensional interacting Bose gas is presented. In the lowest order approximation we obtain the critical temperature $T_c \approx 2\pi n/[m \log(2\pi/mg)]$, where n is the density, m the mass, and g the coupling. This result is based on a new formulation of interacting gases at finite density and temperature which is reminiscent of the thermodynamic Bethe ansatz in one dimension. In this formalism, the basic thermodynamic quantities are expressed in terms of a pseudo-energy. Consistent resummation of 2-body scattering leads to an integral equation for the pseudo-energy with a kernel based on the logarithm of the exact 2-body S-matrix.

1. Introduction

The properties of interacting Bose gases can be very different depending on the spatial dimension. This has become especially interesting in recent years due to the possibility of experimentally realizing lower-dimensional cold gases with magnetic and optical traps. In 3 dimensions there is a critical point for Bose–Einstein condensation (BEC) even in the non-interacting theory, with a critical temperature $T_c = 4\pi(n/\zeta(3/2))^{2/3}$ where n is the density and ζ Riemann's zeta function. In two dimensions the same formula becomes $T_c = 4\pi n/\zeta(1)$ and since $\zeta(1)$ diverges there is no critical point at finite temperature for the non-interacting gas. It is believed however that the two-dimensional interacting gas has a critical point in the universality class of

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the Kosterlitz–Thouless transition rather than BEC [1–4]. (For a review see [5].) This transition has recently been observed in experiments [6].

In the one-dimensional case the particles effectively behave as fermions, the model is integrable, and an exact solution is known based on the thermodynamic Bethe ansatz (TBA) [7, 8]. The one-dimensional case illustrates the importance of non-perturbative methods like the TBA for understanding the physics. Whereas the usual perturbative finite-temperature Feynman diagram method involving Matsubara frequencies in loops is a standard approach which entangles zero temperature perturbation theory with quantum statistical sums, the TBA represents an entirely different organization of the free energy which essentially disentangles the two. More specifically, the only property of the model it is based on is the exact 2-body scattering matrix computed to all orders in perturbation theory at zero temperature. In principle a TBA-like organization of the free energy is possible for non-integrable theories in any dimension based on the formula in [9] which expresses the partition function in terms of the exact S-matrix. The derivation of the TBA as given by Yang and Yang however was not based on the result in [9] but rather relied on the factorizability of the multi-particle S-matrix into 2-body S-matrices for integrable systems. For non-integrable systems the formalism in [9] can be quite complicated since the free energy contains N-body terms which do not factorize. Nevertheless, for a dilute gas the consistent resummation of the 2-body scattering terms can represent a useful approximation that is intrinsically different from other methods.

In this work we derive explicit expressions for the free-energy, occupation numbers, etc., in this 2-body approximation for non-integrable models in any spatial dimension. The result is TBA-like: everything is expressed in terms of a pseudo-energy that satisfies an integral equation with a momentum-dependent kernel which is related to a matrix element of the logarithm of the S-matrix. Our analysis builds on the previous work [10], and the final result presented here contains several important technical improvements.

Most of this paper is devoted to developing the formalism in generality. In Section 3 we describe contributions to the free energy with diagrams, not to be confused with finite temperature Feynman diagrams, where the vertices represent N-body interactions for any N. The cluster decomposition of the S-matrix is necessary to establish the extensivity of the free energy. In Section 4 we derive an integral equation for the occupation numbers (filling fractions) from a variational principle based on a Legendre transformation that exchanges the chemical potential with the filling fraction. In Section 5 we present the integral equation that consistently resums the infinite number of 2-body diagrams. In Section 6 the 2-body kernel that appears in the integral equation is derived for interacting Bose gases in 2 and 3 dimensions. In Section 7 we compare our approximation to the exact TBA for the one-dimensional case.

This new formalism is illustrated in the two-dimensional case. Since the 2-body approximation is reasonably simple in its final form, we present a self-contained analysis of the critical point of the Bose gas in the next section, where we derive an expression for the coupling constant dependent critical density.

2. Critical point of the two-dimensional Bose gas

In this section we illustrate the main ingredients of our formalism by applying it to the critical point of the two-dimensional Bose gas. Our treatment is significantly different from previous ones [1–4]. Although we find results that are similar to the known results, we find our treatment to be considerably simpler and transparent and doesn't rely on an effective description of vortices.