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Equivariant reduction of Yang–Mills theory over the fuzzy sphere and the emergent vortices

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Abstract

We consider a U(2) Yang–Mills theory on $\mathcal{M} \times S_F^2$ where $\mathcal{M}$ is a Riemannian manifold and S_F^2 is the fuzzy sphere. Using essentially the representation theory of SU(2) we determine the most general SU(2)-equivariant gauge field on $\mathcal{M} \times S_F^2$. This allows us to reduce the Yang–Mills theory on $\mathcal{M} \times S_F^2$ down to an Abelian Higgs-type model over $\mathcal{M}$. Depending on the enforcement (or non-enforcement) of a "constraint" term, the latter may (or may not) lead to the standard critically-coupled Abelian Higgs model in the commutative limit, $S_F^2 \to S^2$. For $\mathcal{M} = \mathbb{R}^2$, we find that the Abelian Higgs-type model admits vortex solutions corresponding to instantons in the original Yang–Mills theory. Vortices are in general no longer BPS, but may attract or repel according to the values of parameters.

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1. Introduction

It is commonplace in modern physics to consider field theories defined on manifolds of the form $\mathcal{M} \times X$, where $\mathcal{M}$ represents physical space and X is some compact manifold. One popular example is to consider pure Yang–Mills theory, with X a coset space G/H. In this case the group G acts naturally on its coset; by requiring the gauge fields to be invariant under the action of G

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up to a gauge transformation, one obtains a new gauge theory on $\mathcal{M}$. In this way a relatively complicated theory on $\mathcal{M}$ is obtained from a relatively simple theory on $\mathcal{M} \times X$. We shall call such a process "equivariant reduction".

The first example of equivariant reduction was due to Witten [1]. He showed that Yang–Mills theory on $\mathbb{R}^4$ reduces under SU(2)-equivariance to an Abelian Higgs model on a 2-dimensional hyperbolic space $\mathbb{H}^2$, and thereby constructed the first instantons with charge greater than 1. The space $\mathbb{H}^2$ emerges naturally in this example, because $\mathbb{R}^4 \backslash \mathbb{R}^2$ is conformal to $\mathbb{H}^2 \times S^2$, and Yang–Mills theory is conformally invariant in four dimensions.

In subsequent years two major formalisms have been developed to perform more exotic equivariant reductions. Historically, the first was "coset space dimensional reduction" (CSDR) [2,3], which uses intrinsic coordinates on the coset space, and is generally used as a method to try to obtain the standard model on the Minkowski space $\mathcal{M} = M^4$ starting from a Yang–Mills–Dirac theory on the higher dimensional space $M^4 \times G/H$. The second is the "quiver" approach [4–7], which uses a more sophisticated language of equivariant vector bundles, and has the interesting feature of reducing self-dual instantons on $\mathcal{M} \times X$ to BPS vortices on $\mathcal{M}$. The two approaches seem on the whole to be equivalent, but tend to emphasise different features of equivariant reduction. In particular, Witten's example is the basic one in both approaches.

The quiver approach has also been applied to the case where $\mathcal{M}$ is a non-commutative manifold (the $2d$-dimensional Moyal space $\mathbb{R}^{2d}_\theta$) and with some success: the dimensionally reduced Bogomolny equations are, for appropriate choice of parameters, integrable [4]. So it is natural to ask: what happens when the coset space X, instead of the physical space $\mathcal{M}$, is non-commutative, or both spaces are non-commutative? In particular, does the reduced theory still have vortices, and are they BPS? In this paper, we will focus on the case, where only the coset space X is non-commutative.

A particular class of non-commutative coset spaces have been known for quite some time in the literature. Namely, these are the "fuzzy spaces", of which the simplest and the most famous example is the fuzzy sphere, S_F^2 [8,9]. Gauge theory has been formulated on S_F^2 [10–12] and the group SU(2) acts naturally on it, so it seems well-suited for equivariant reduction. Actually, equivariant reduction over fuzzy spaces has already been discussed in the literature, using the CSDR approach [13]. However, only very simple examples have been studied so far, and not in great detail, so it seems important to try to perform an equivariant reduction in full. In particular, one should compare equivariant reduction over fuzzy spaces with reduction over normal coset spaces to see what new features emerge. It is worth mentioning that the fuzzy sphere appears in other gauge-theoretic contexts, such as the Aharony–Bergman–Jafferis–Maldacena model [14]. Equivariant reduction might prove a useful tool for constructing solutions to such models, perhaps along the lines of [15].

With these motivations in mind, in this paper we present the fuzzy generalisation of Witten's equivariant reduction over $\mathcal{M} \times S^2$. To this end, we start from a U(2) Yang–Mills theory on $\mathcal{M} \times S_F^2$ and using essentially the representation theory of SU(2) we determine the most general SU(2)-equivariant gauge field on $\mathcal{M} \times S_F^2$. This allows us to compute the reduced action in full. The latter appears to be an Abelian Higgs-type model over $\mathcal{M}$. Specializing to a concrete and a simple case by selecting $\mathcal{M} = \mathbb{R}^2$, we demonstrate that this model admits classical vortices and present their numerical solutions.

An outline of the rest of this paper is as follows: in Section 2 we will review gauge theory on $\mathcal{M} \times S_F^2$, in particular emphasising the approach in which it can be dynamically generated by a gauge theory on $\mathcal{M}$ with a larger gauge group. In Section 3 we will review equivariant reduction over the fuzzy sphere, and give an explicit parametrisation of the equivariant gauge fields. In