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$(Z\alpha)^4$ order of the polarization operator in Coulomb field at low energy

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Abstract

We derive the low-energy expansion of $(Z\alpha)^2$ and $(Z\alpha)^4$ terms of the polarization operator in the Coulomb field. Physical applications such as the low-energy Delbrück scattering and "magnetic loop" contribution to the g factor of the bound electron are considered.

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1. Introduction

One of the predictions of the quantum field theory is a vacuum polarization by an external field. An important case thoroughly studied both experimentally and theoretically is the vacuum polarization effects in atomic field. Methods used for the study of this effect essentially depend on the nuclear charge $Z|e|$. At low Z, the perturbation theory with respect to $Z\alpha$ is applicable ($\alpha = e^2 = 1/137$ is the fine structure constant,[1] $\hbar = c = 1$). At high Z, the interaction with the external field should be taken into account exactly, which can be done with the help of the electron Green function in this field. This approach often requires quite involved numerical calculations, which usually fail to give the results for low Z. Thus, the two approaches tend to be complementary.

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[1] The fine structure constant is expressed in the Gaussian CGS system instead of the Heaviside one where $\alpha = e^2/(4\pi)$.

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Usually, the perturbative calculations of vacuum polarization effect are limited by the leading order, since the first nonvanishing correction involves two more loops. Nowadays, the modern methods of calculation of the multiloop integrals are sufficiently powerful for the calculation of higher orders in $Z\alpha$. It provides a possibility to compare the results of these approaches.

One of the basic nonlinear QED processes in the atomic field is the Delbrück scattering [1], the scattering of the photon in the Coulomb field due to the vacuum polarization. The amplitude of this process in the Born approximation has been obtained long ago for arbitrary energies in Ref. [2]. At high energies and small scattering angles, when the quasiclassical approximation is valid, the amplitude is known exactly in $Z\alpha$, see Refs. [3–8]. Recently in Ref. [9], the Delbrück amplitude has been calculated numerically exactly in the parameter $Z\alpha$ at low energies. It was shown that the contribution of the higher orders (Coulomb corrections) to the amplitude can be well fitted by the polynomial $C_4(Z\alpha)^4 + C_6(Z\alpha)^6$. The calculation of $(Z\alpha)^4$ term in perturbation theory would provide the independent check of the result of Ref. [9].

In present paper, we consider the polarization operator $\Pi^{\mu\nu}(\omega, \mathbf{k}, \mathbf{q})$ in the Coulomb field for small external momenta

$$\omega \sim |\mathbf{k}| \sim |\mathbf{q}| \sim \lambda m, \tag{1}$$

where λ is a dimensionless small parameter,

$$\lambda \ll 1. \tag{2}$$

We calculate the expansion of the polarization operator in the Coulomb field in λ and $Z\alpha$ up to the order $\lambda^4(Z\alpha)^4$. The low-energy Delbrück scattering amplitude is readily expressed in terms of this operator. This polarization operator is also an essential ingredient of calculations of different physical observables in atoms, like Lamb shift and magnetic moment of the bound particle.

The polarization operator in the external Coulomb field is determined as follows:

$$\Pi^{\mu\nu}(\omega, \mathbf{k}, \mathbf{q}) = 4\pi i e^2 \int d\mathbf{x}\, d\mathbf{y}\, dt\, e^{-i\omega t + i\mathbf{kx} - i\mathbf{qy}} \langle \mathrm{vac}|\mathrm{T} J^\mu(t, \mathbf{x}) J^\nu(0, \mathbf{y})|\mathrm{vac}\rangle, \tag{3}$$

where $J^\mu = \bar\psi\gamma^\mu\psi$ is the electron current and the state $|\mathrm{vac}\rangle$ corresponds to the vacuum state in the presence of the Coulomb potential $Z|e|/r$. In e^2 order, we have

$$\Pi^{\mu\nu}(\omega, \mathbf{k}, \mathbf{q}) = 4\pi i e^2 \int \frac{d\varepsilon}{2\pi} d\mathbf{x}\, d\mathbf{y}\, e^{i\mathbf{kx} - i\mathbf{qy}} \,\mathrm{Tr}\big[\gamma^\mu G(\mathbf{x}, \mathbf{y}|\varepsilon - \omega)\gamma^\nu G(\mathbf{y}, \mathbf{x}|\varepsilon)\big], \tag{4}$$

where $G(\mathbf{y}, \mathbf{x}|\varepsilon)$ is the Green function of the electron in the Coulomb field. Due to the gauge invariance, the polarization operator obeys the constraints

$$k_\mu \Pi^{\mu\nu}(\omega, \mathbf{k}, \mathbf{q}) = q_\nu \Pi^{\mu\nu}(\omega, \mathbf{k}, \mathbf{q}) = 0, \tag{5}$$

where $k^0 = q^0 = \omega$. Using these constraints, we can express $\Pi^{\mu\nu}$ via five independent tensor structures, which we choose as follows:

$$\begin{aligned} m^3\Pi^{\mu\nu} = {}& f_1\big(g^{\mu\nu} k\cdot q - q^\mu k^\nu\big) - f_3 \epsilon^{\mu\alpha\beta\gamma} n_\alpha \epsilon^{\nu\rho\sigma\tau} n_\rho \frac{k_\beta (k-q)_\gamma (k-q)_\sigma q_\tau}{(k-q)^2} \\ & + \big(n^\mu k^\alpha - \omega g^{\mu\alpha}\big)\big(q^\beta n^\nu - \omega g^{\beta\nu}\big)\left[f_2 g_{\alpha\beta} + f_4 \frac{k_\alpha q_\beta}{\omega^2} - f_5 \frac{(k-q)_\alpha (k-q)_\beta}{(k-q)^2}\right], \end{aligned} \tag{6}$$

where $n = (1, \mathbf{0})$ and f_i are some scalar functions of ω, $\mathbf{k}$ and $\mathbf{q}$.