Available online at www.sciencedirect.com
ScienceDirect
Nuclear Physics B 810 [FS] (2009) 527–541
NUCLEAR PHYSICS B
www.elsevier.com/locate/nuclphysb

Quantization of sine-Gordon solitons on the circle: Semiclassical vs. exact results

Michael Pawellek

Institut für Theoretische Physik III, Universität Erlangen–Nürnberg, Staudtstr. 7, D-91058 Erlangen, Germany

Received 14 August 2008; accepted 1 October 2008

Available online 6 October 2008

Abstract

We consider the semiclassical quantization of sine-Gordon solitons on the circle with periodic and anti-periodic boundary conditions. The 1-loop quantum corrections to the mass of the solitons are determined using zeta function regularization in the integral representation. We compare the semiclassical results with exact numerical calculations in the literature and find excellent agreement even outside the plain semiclassical regime.

PACS: 03.65.Sq; 11.15.Kc; 11.27.+d

Keywords: Sine-Gordon theory; Semiclassical quantization; Zeta function regularization; Lamé equation

1. Introduction

The semiclassical quantization method is a fruitful technique to explore non-perturbative properties of quantum field theories [1–3]. The determination of quantum corrections to the mass of the ϕ^4 kink and the sine-Gordon soliton are classic textbook examples of this method [4]. Although the sine-Gordon model is integrable [5] and the ϕ^4 model is not [6], on the semiclassical level these theories are surprisingly similar in the one soliton/kink sector. In both cases the fluctuation equation obtained after expansion around the classical soliton/kink solutions are exactly solvable reflectionless Schrödinger equations of the Pöschl–Teller type [4]. Considering these models on a compact space, e.g., a circle, one gets instead *quasi-exactly* solvable [7] fi-

E-mail address: michi@theorie3.physik.uni-erlangen.de.

0550-3213/$ – see front matter
doi:10.1016/j.nuclphysb.2008.10.001

nite gap Schrödinger equations of the Lamé type [8,9]. In these cases only a finite number of (anti-)periodic eigenfunctions and eigenvalues can be determined analytically [10].

For quantum corrections to the energy of non-trivial field configurations one needs at first sight information of the full fluctuation spectrum. In [11] the special finite-gap properties of the $n = 2$ Lamé equation [10] and the integral representation of the spectral zeta function [12–14] were used to construct an analytic result for the 1-loop quantum corrections to the mass of the twisted ϕ^4 kink on the circle without explicit knowledge of the spectrum. It was found that an energetically preferred radius exists, where the contributions of the classical and 1-loop part are of the same order of magnitude. Therefore the question arises if higher loop corrections may spoil this picture.

To settle this question we will not consider higher loop effects directly, but take a different route. We will consider the sine-Gordon model on S^1, since it is very similar in the semiclassical approximation to the ϕ^4 theory. The fluctuation equation of (anti-)periodic solitons on S^1 of the sine-Gordon model is the $n = 1$ Lamé equation [8]. Therefore we can apply the techniques used for the ϕ^4 model [11] also in this case. The integrability enables us to compare semiclassical results with exact results of the soliton energy, obtained in [15,16] by numerically solving the corresponding non-linear integral equations (NLIE) [17]. This will give new insights into the question on relevance of higher loop corrections on S^1 for the sine-Gordon soliton.

We will concentrate in the following on the one soliton sector on the compact manifold S^1, where we can impose two different boundary conditions:

- Periodic b.c.: $\phi(x + R) = \phi(x) + \frac{2\pi}{\beta}$;
- Anti-periodic b.c.: $\phi(x + R) = -\phi(x) + \frac{2\pi}{\beta}$.

In the past only asymptotic expressions of the semiclassical 1-loop energy for $k \to 0$ and $k \to 1$ of the elliptic modulus were obtained [8,9,18]. We will give analytic results valid for all k and therefore R.

First we review the classical solutions [8] of the corresponding b.c. and then use the spectral discriminant of the $n = 1$ Lamé equation [9] to determine the 1-loop contributions. Finally we compare our result with numerical calculations, which used the integrability of sine-Gordon model [15,16].

2. Classical solutions

We consider the sine-Gordon model with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi), \qquad V(\phi) = \frac{m^2}{\beta^2}\big(1 - \cos(\beta\phi)\big) \tag{1}$$

with spatial direction compactified on a circle with circumference R. We review the properties of classical static solutions [8,18] of the equation of motion following from (1) on S^1.

2.1. Periodic boundary condition

With (quasi-)periodic boundary conditions $\phi(x + R) = \phi(x) + \frac{2\pi}{\beta}$ the static field configuration is given by

$$\phi_0(x) = \frac{\pi}{\beta} + \frac{2}{\beta}\,\mathrm{am}\left(\frac{m(x - x_0)}{k}, k\right), \tag{2}$$