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The scaling window of the 5D Ising model with free boundary conditions

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Abstract

The five-dimensional Ising model with free boundary conditions has recently received a renewed interest in a debate concerning the finite-size scaling of the susceptibility near the critical temperature. We provide evidence in favour of the conventional scaling picture, where the susceptibility scales as L^2 inside a critical scaling window of width $1/L^2$. Our results are based on Monte Carlo data gathered on system sizes up to $L = 79$ (ca. three billion spins) for a wide range of temperatures near the critical point. We analyse the magnetisation distribution, the susceptibility and also the scaling and distribution of the size of the Fortuin–Kasteleyn cluster containing the origin. The probability of this cluster reaching the boundary determines the correlation length, and its behaviour agrees with the mean field critical exponent $\delta = 3$, that the scaling window has width $1/L^2$.

1. Introduction

The Ising model in dimension $d = 5$ is of particular interest since it is the first case where the model is strictly above its upper critical dimension $d_c = 4$. Rigorous results [1,2] establish that the critical exponents of the model assume their mean field values. Here the specific heat exponent $\alpha = 0$, and the results of [2] also imply that the specific heat is bounded at the critical

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point. Various other properties have also been shown to have mean-field behaviour, and a unified proof of this was given by Sakai [3] who developed a working version of the lace expansion for the Ising model. Additionally, recent simulation results [4] indicate that, just as for the mean field case, the specific heat is discontinuous at the critical point.

In contrast to these asymptotic results there has been a long running debate over the finite size scaling for the model with free boundary conditions. We'll refer the reader to [5] for a fuller overview of the history and stick to the presently most relevant parts here. For $d = 5$ and cyclic boundary conditions there is agreement that e.g. $\chi \propto L^{5/2}$ for a lattice of side L. The conventional picture for the free boundary case is that $\chi \propto L^2$. However, it has also been suggested [5] that the free boundary case should scale in the same way as the cyclic boundary case near the finite size susceptibility maximum. A computational study [6] of the, then, largest lattices possible supported the conventional picture, but in [5] it was suggested this was due to an underestimate of the influence of the large boundaries of the used systems. For systems exactly at the critical coupling this issue was resolved in [7] where a study of systems up to $L = 160$ demonstrated an increasingly good agreement with the conventional picture as the system size was increased. But, this left the behaviour in the rest of the critical scaling window open.

The aim of the current paper is to extend the study of large systems from [7] to the full critical window, including the location of the maximum of χ, and give the best possible estimates for the scaling behaviour in the coupling region discussed by all the previous papers. Apart from the susceptibility we also study properties of the Fortuin–Kasteleyn cluster containing the origin and use those to estimate both the susceptibility and the correlation length of the model.

To concretize, the predictions from [5] are that the location of the maximum for χ will scale as L^{-2} and the maximum value as $L^{5/2}$. The more recent [8] agrees with these predictions, and also the prediction from [9] that the location of the maximum of the susceptibility should scale as L^{-2}, as does [10], but both are based on smaller system sizes than those considered in the present work.

In short, our conclusion is that the data is well fitted by the conventional scaling picture, both for the location and value of the susceptibility, and location of the finite size critical point for the magnetisation.

2. Definitions and details

For a given graph G on N vertices the Hamiltonian with interactions of unit strength along the edges is $\mathcal{H} = -\sum_{ij} s_i s_j$ where the sum is taken over the edges ij. Here the graph G is a 5-dimensional grid graph of linear order L with free boundary conditions, i.e. a cartesian product of 5 paths on L vertices, so that the number of vertices is $N = L^5$ and the number of edges is $5L^5(1 - 1/L)$. We use $K = 1/k_B T$ as the dimensionless inverse temperature (coupling) and denote the thermal equilibrium mean by $\langle \cdots \rangle$. The critical coupling K_c was recently estimated by us [4] to $K_c = 0.11391498(2)$. We will define a rescaled coupling as $\kappa = L^2(K - K_c)$ which gives a scaling window of width L^{-2}. The standard definitions apply; the magnetisation is $M = \sum_i S_i$ (summing over the vertices i) and the energy is $E = \sum_{ij} S_i S_j$ (summing over the edges ij). We let $m = M/N$, $U = E/N$ and $\mathcal{U} = \langle U \rangle$.

Generally our terminology follows that of e.g. [11], and here we explicitly state the definitions most used in this paper. The susceptibility is $\chi = \langle M^2 \rangle / N$ while we define the modulus susceptibility as $\bar{\chi} = \mathrm{var}(|M|)/N$. The standard deviation is denoted σ, as is customary. We will refer to the point where the distribution of M changes from unimodal to bimodal as $K_c^*(L)$, or, in its